# Math Fundamentals for Statistics (Math 52)

# Unit 3: Addition and Subtraction

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“The ‘How’ and ‘Whys’ Guys”

# 3.1: Place Value (Addition Preview)

Our system is a base-ten, place value system. This means that a number like 21 has a different meaning than 12. A base-ten system means that each value moving from right to left is worth 10 times the previous amount. The digit 2 in the number 2 is worth 2 "ones" while the digit 2 in the number 20 is worth 2 "tens", which is 10 times the original value. The digit 2 in the number 200 is worth 2 "hundreds", which is 10 times 20.

Our place value system does go in both directions of the decimal place. Numbers to the right of the decimal point are known as 'decimals.' Notice that numbers to the left of the decimal point are grouped into clumps of three place values – each has hundreds, tens, and ones. When writing a number out, these groups are separated by commas to make it easier to read.

The decimal point is the starting point for all numbers we read and write.

| Billions | | | Millions | | | Thousands | | | Ones | | | | Decimals | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Hundred billions | Ten billions | Billions | Hundred millions | Ten millions | Millions | Hundred thousands | Ten thousands | Thousands | Hundreds | Tens | Ones | ← Decimal Point | Tenths | Hundredths | Thousandths | Ten-thousands | Hundred-thousandths |
| | 6 | 8 | 2 | 4 | 1 | 0 | 3 | 9 | 0 | 0 | 5 | . | 2 | 1 | 7 | | |

Think about the names of the numbers and how we could use different words to represent the same quantity. As an example, if you were buying something for $1,400, what words would you use to represent the value?

The most common are probably "fourteen hundred" and "one thousand, four hundred." Maybe you can even think of the cash you'd bring to pay for the item: 14 one hundred dollar bills would represent "fourteen hundred," while 1,400 one dollar bills would represent "one thousand, four hundred."

14,000 could be thought of in the following ways (and each is helpful at different times):

- 14,000 ones (when counting by ones)
- 1,400 tens (counting by tens)
- 140 hundreds (counting by hundreds)
- 14 thousands (counting by thousands)
- 1.4 ten thousand
- 0.14 hundred thousand
- 0.014 million
- 140,000 tenths

So you can see here, that there are many ways to write the same number – a handy skill to possess!

**EXPLORE!** Let's take this in a different direction – what are two different ways we could describe the following values but using different names:

A) ** 9,800

C) 460,000

B) 12,000

D) 5,000,000

**EXPLORE!** With this in mind, re-write the following in standard notation. "16 tens" would be written as 160 in standard place value notation.

A) ** 1.4 tens

E) 1.25 million

B) 0.037 ten thousand

F) 689 thousandths

C) 3.89 hundreds

G) 31 tenths

D) 48 thousands

H) 3,045 hundredths

# 3.2: Addition

The key concept with addition is combining and counting the number of like objects. One basic example is with office furniture. With addition, there are two **addends** and the result of the operation is called the **sum**. In $a + b = c$, the addends are $a$ and $b$, while the sum is $c$.

Example: Imagine you're doing inventory at a store and write out the following list…
3 chairs and 7 tables and 8 chairs and 4 chairs. How could we re-write this in a simpler way?

3 chairs + 7 tables + 8 chairs + 4 chairs = 15 chairs + 7 tables.

Objects that are the same can be spotted and combined quickly, but remember, they must be exactly the same – an orange is not an apple, even though both are pieces of fruit!

**EXPLORE!** Try a few on your own to add like terms. 🖩 means use your calculator!

A) 4 hats + 3 coats + 2 coats + 11 hats + 6 coats =

B) 🖩 147 apples + 63 oranges + 97 apples + 23 oranges + 184 oranges + 453 apples =

C) 5 negatives + 11 positives + 3 negatives + 8 negatives + 2 positives =

D) $5x + 3y + 10x + 27x + 19y =$

E) $2\sqrt{26} + 7\sqrt{15} + 7\sqrt{26} + 9\sqrt{15} + \sqrt{15} =$

F) 3 sevenths + 11 fifths + 2 sevenths + 3 fifths =

G) 🖩 983 euro + 574 yen + 1491 euro + 374 yen + 9473 yen =

H) $12A + 4B + 7C + 8D + 9B + 4D + 11C + 3B + 5D + 23A =$

I) $16\pi + 37\alpha + 19\theta + 29\pi + 18\theta + 61\alpha + 11\alpha =$

J) 11 hundreds + 87 tens + 6 hundreds + 11 tens + 3 ones =

Another skill that is quite important is to be able to rewrite sums as the original addends. As an example, we typically see the problem 14 + 23 = 37. How could you re-write this equation going the other direction?  [Hint: there could be more than one answer!]  37 = 14 + 23 would be one way to rewrite 37 using addends, but 37 = 30 + 7 is another. Rewriting a number into a sum is known as **regrouping**.

Example: Find at least 5 ways to rewrite the number 23 using regrouped addends:

23 = 20 + 3     23 = 21 + 2     23 = 15 + 8     23 = 19 + 4     23 = 22 + 1

Interactive Example: Find at least 4 ways to rewrite the number 74 using regrouped addends:

**EXPLORE!** Try a few on your own; rewrite using regrouped addends.

A) (at least two ways) 48 =

B) (at least two ways) 61 =

C) 🖩 (at least three ways) 632 =

D) (at least three ways) 100 =

**EXPLORE!** When adding whole numbers like 41 + 238, we typically line up the place values. Explain why we do this?

In order to be able to do addition more quickly, mathematicians create an **algorithm** – a process to follow that will produce a guarantee result. Many of you already know some algorithms, but we will introduce a few different algorithms. When you encounter a problem, having more than one set of tools is important so that you can determine which technique will be the most efficient.

**Algorithm #1:** Partial Sums

In this algorithm, we combine numbers based on their place values. When necessary, we'll regroup the values to be in a higher place value. Here's an example using: $536+176$.

*Step 1:* Write the problem and line up the place values and draw a line underneath.

*Step 2:* Starting with the ones place value, add the digits and write the sum underneath the line. Because we're only adding part of the numbers, we call this a partial sum. 6 + 6 = 12

*Step 3:* Move to the next place value (tens), add the digits and write the sum underneath the line. Here 3 tens + 7 tens = 10 tens, or 100.

*Step 4:* Move to the next place value (hundreds), add the digits and write the sum underneath the line. 5 hundreds + 1 hundred = 600.

*Step 5:* When finished, draw a line under the last partial sum and then add these partial sums and the result is the sum of the original addends. So $536+176=712$.

| ***Step 1*** | ***Step 2*** | ***Step 3*** | ***Step 4*** | ***Step 5*** |
|---|---|---|---|---|
| 536<br>+176 | 536<br>+176<br>12 | 536<br>+176<br>12<br>100 | 536<br>+176<br>12<br>100<br>600 | 536<br>+176<br>12<br>100<br>600<br>712 |

$$\begin{array}{rl} 536 & \leftarrow \text{Addend} \\ \underline{+176} & \leftarrow \text{Addend} \\ 12 & \leftarrow \text{Partial Sum of Ones} \\ 100 & \leftarrow \text{Partial Sum of Tens} \\ \underline{600} & \leftarrow \text{Partial Sum of Hundreds} \\ 712 & \leftarrow \text{Final Sum} \end{array}$$

Here you can see all of the parts in order with their names.

**Algorithm #2:** Traditional Algorithm

In this algorithm, we essentially use the partial sum algorithm, and then just do the partial sums and add them together inside the process.

*Step 1:* Write the problem and line up the place values and draw a line underneath.

*Step 2:* Starting with the ones place value, add the digits. With this partial sum, we regroup if necessary – any partial sum greater than 9 will be written as tens and ones. In our problem, 6 + 6 = 12, which is 1 ten and 2 ones. We will write the 1 group of ten above the other tens (above the line), and write the 2 ones in the ones place below the line.

*Step 3:* Move to the next place value (tens), add the digits and any regrouped values together. Once you have the sum, regroup into tens and hundreds if possible. Here 3 tens + 7 tens + 1 ten (regrouped) = 11 tens, or 1 hundred and 1 ten. We will write the 1 group of hundred above the other hundreds (above the line), and write the 1 ten in the tens place below the line.

*Step 4:* Move to the next place value (hundreds), add the digits and write the sum underneath the line. Move to the next place value (hundreds), add the digits and any regrouped values together. 5 hundreds + 1 hundred +1 hundred (regrouped) = 7 hundreds. In this case, there is nothing left to regroup. We will write the 7 groups of hundred in the hundreds place below the line. With no additional place values, the algorithm is finished showing $536+176=712$.

| ***Step 1*** | ***Step 2*** | ***Step 3*** | ***Step 4*** |
|---|---|---|---|
| $\begin{array}{r} 536 \\ \underline{+176} \end{array}$ | $\begin{array}{r} {\scriptstyle 1}\phantom{6} \\ 536 \\ \underline{+176} \\ 2 \end{array}$ | $\begin{array}{r} {\scriptstyle 1\,1}\phantom{6} \\ 536 \\ \underline{+176} \\ 12 \end{array}$ | $\begin{array}{r} {\scriptstyle 1\,1}\phantom{6} \\ 536 \\ \underline{+176} \\ 712 \end{array}$ |

If you look closely, you can see the partial sums when we write this algorithm out. This one is named the traditional algorithm as it is typically taught to students in the US. Often, this is the only algorithm used… but it is not the only algorithm that exists. For many people, they find this algorithm more efficient than the partial sums because there is less to write.

**For Love of the Math:** *The language often used historically when teaching this algorithm is "carrying." Because the word carrying doesn't indicate where the numbers are being carried to and really doesn't demonstrate the actual concept, the terminology has been changed. Instead, we now call this process "regrouping." Instead of keeping 12 written this way, we write it as 10 + 2.*

**Algorithm #3:** Rewriting addends (compatible numbers)

Example 1: In this algorithm, we rewrite the addends to make them easier to compute. This process, when done well, can be done mentally… and quickly. Let's do **536 + 176**.

*Step 1:* Look at the two numbers and consider what it would take to make the numbers easier. For the number 536, it would be nicer if there were 4 more ones. So we just rewrite 176 to make it easier. [If the sum is quick, just do the sum.] $536+(176)=536+(4+172)=(536+4)+172=540+172$.

*Step 2:* Now consider the new value, 540. It would be nicer if there were 6 more tens. So we just rewrite 172 to make it happen. $540+(172)=540+(60+112)=(540+60)+112=600+112$.

*Step 3:* Once the problem is made easier, and there are no more regroupings, you can add the resulting values very quickly. $600+112=712$.

Example 2: For this algorithm, another example could be helpful. We'll try **847 + 625**.

*Step 1:* There are enough ones to regroup, so we'll adjust to make some into tens.
$847+(625)=847+(3+622)=(847+3)+622=850+622$

*Step 2:* Once we look at the result, the ones and tens would add quickly. In fact, so would the hundreds since 8 hundreds + 6 hundreds = 14 hundreds. $850+622=1472$.

Example 3: For smaller numbers, it may be quicker to add the larger values and work your way smaller. Here's one to show that: **428 + 65**.

*Step 1:* Start with the 428 and think of adding 60 then add 5:
$428+65=428+60+5=488+5=488+2+3=490+3=493$

Example 4: For numbers very close to groups of ten or hundred, this process can make the addition much quicker. Last example: **427 + 398**.

*Step 1:* See how the 398 is very close to 400, and is off by only 2. If we rewrite the 427 to use a 2, then we'll be able to find the sum quickly! $427+398=425+2+398=425+400=825$. This one can be really, really fun and extremely quick!

Interactive Example: Use this last algorithm on: **2,248 + 699**. Try to do the sum mentally.

**EXPLORE!** Determine which algorithm is the best (most efficient) one to use on the following:

| | | Partial Sums | Traditional | Regroup Addends |
|---|---|---|---|---|
| A) ** | $468+27$ | | | |
| B) | $843+596$ | | | |
| C) | $1{,}843+2{,}596$ | | | |
| D) | $46{,}283+21{,}943$ | | | |
| E) | $785+439$ | | | |
| F) | $6{,}385+5{,}429$ | | | |

**EXPLORE!** Use any of the algorithms shown to practice doing addition. Rewrite the following and add vertically – regroup when necessary:

A) ** $468+27$

B) $843+596$

C) $1{,}843+2{,}596$

D) $46{,}283+21{,}943$

E) $785+439$

F) $6{,}385+5{,}429$

# 3.3: Addition Properties

There are some interesting properties about addition with numbers that will help us in many places. However, in order to determine where the properties come from, some examples can help.

**EXPLORE!** Based on our addition knowledge, find the following values:
(Split room – Left side = LS, Right side = RS)

A) 7 + 9 (LS)

C) 231 + 519 (LS)

B) 9 + 7 (RS)

D) 519 + 231 (RS)

What do you notice about addition and the *order* of the addends? How could you rewrite an expression like $a + b$?

This general rule is called the **Commutative Property of Addition:** $a+b=b+a$ for all numbers. You might think of this like it was a commute to work and then home – it is the same distance from work to home as it is from home to work. Likewise, changing the order of the addends will not change the sum. Actually, when you think about it, the fact that there is only one word for both values in addition indicates that this property will hold true!

**EXPLORE!** Let's try another one based on our addition knowledge, find the following values:

A) 2 + (3 + 4) (LS)

C) (2 + 9) + 13 (LS)

B) (2 + 3) + 4 (RS)

D) 2 + (9 + 13) (RS)

What do you notice about *grouping* of the addends? How could you rewrite an expression like $(a+b)+c$?

This general rule is called the **Associative Property of Addition:** $(a+b)+c=a+(b+c)$ for all numbers. You might think of this like it was about the grouping – changing the grouping of the addends will not change the sum.

**EXPLORE!** There are a few other properties that might seem obvious, but are often helpful:

A) 0 + 9

B) 231 + 0

This general rule is called the **Identity Property of Addition:** $a + 0 = 0 + a = a$ for all numbers. Anytime we add 0 to a number, or a number to 0, the result is just the original number. For addition, the number zero has a very special place in mathematics and is called the **Additive Identity Element**. One way to picture this property is to think that each number has an identity, like people do. When you add 0 to it, the identity or value doesn't change.

**EXPLORE!** Now that we know some of the properties, let's see if we can identify them in action. Determine which property is being used in each step. [There may be more than one!]

- Commutative Property of Addition: $a + b = b + a$
- Associative Property of Addition: $(a + b) + c = a + (b + c)$
- Identity Property of Addition: $a + 0 = 0 + a = a$

A) ** $75 + (378 + 25) = 75 + (25 + 378) = (75 + 25) + 378$

B) $72 + 0 = 72$

C) $16 + (14 + 7) = (16 + 14) + 7$

D) $596 + (32 + 94) = 596 + (94 + 32)$

E) $(284 + 153) + 0 = 284 + (153 + 0) = 284 + 153$

F) $284 + 153 = 153 + 284$

The properties of addition can make it easy to perform addition, as seen in part (A).

# 3.4: Adding Integers using Chips

Integers are more challenging than whole numbers; sometimes having a hands-on model can give a way of visualizing these types of expressions that works better for some people. The next pages show a way to use something like poker chips to demonstrate addition with integers. Sometimes having a concrete example can help with the more abstract concepts.

For these examples, we'll use "B" for black chips and "R" for red chips as this will relate to a business situation, where 'being in the **black**' is a way of saying that things are positive, while 'being in the **red**' means things are negative. As in our definition of addition, the thought of adding two groups would be to put them together.

One thing to notice with the chips is a very important fact: (1) + ( – 1) = 0, so if we ever have one black and one red, B R, we'll call that a **zero pair**. Having one black and one red would be one zero pair.

**EXPLORE!** Write out zero pairs as necessary using numbers and chips. The first is shown for you.

A) Two zero pairs: *numbers:* 2 + (– 2)
*chips:* BR BR

B) ** Four zero pairs

C) Three zero pairs

Now let's examine how we can write numbers in many different ways. The number 3 for example, could be represented by three black chips: BBB. Or we could represent the number 3 in a different way, with four black chips and one red chip: BBBBR. Notice that the last two, BR, form a zero pair. Can you think of another way to represent the number 3?

**EXPLORE!** Represent the number listed using poker chips in at least two ways using zero pairs.

A) ** 1

C) – 1

B) – 3

D) 2

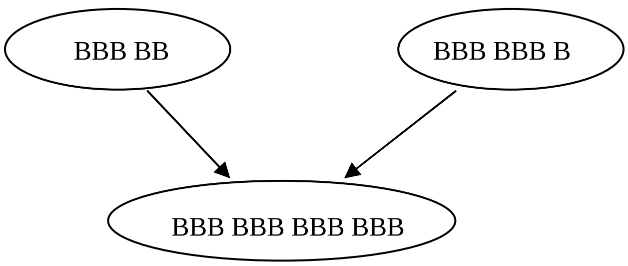
Addition problems with whole numbers and integers can be demonstrated using this model, which reflects the mathematical properties in a more concrete way – sometimes we will find zero pairs, and sometimes we'll need to re-write a number using a different representation. Remember that addition requires combining *like objects*.

Examples:

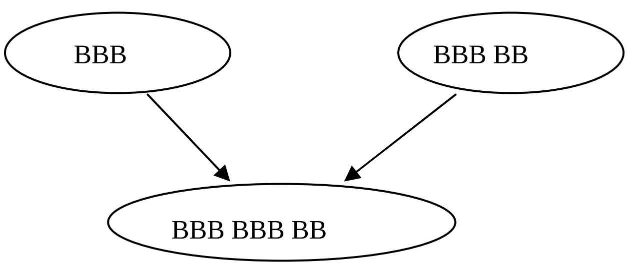
A) $5 + 7$ B) $3 + 5$ C) $(-5) + (-2)$

Here's how to set up with chips:

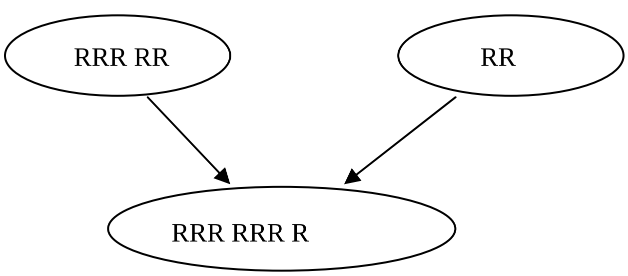
A) First, write out the number 5 (BBB BB) then the number 7 (BBB BBB B), and push the chip piles together – adding like terms. This shows that $5 + 7 = 12$, since we end with 12 black chips. We will end with the following type of picture. Notice how the two numbers are added (or put together)!




B) First, write out the number 3 (BBB) then the number 5 (BBB BB), and push the chip piles together – adding like terms. This shows that $3 + 5 = 8$, since we end with 8 black chips.




C) First, write out the number – 5 (RRR RR) then the number – 2 (RR), and push the chip piles together – adding like terms. This shows that $(-5) + (-2) = -7$, since we end with 7 red chips. We will end with the following type of picture.




For most people, adding whole numbers like these examples might be quicker to do without chips. The chips should be used briefly to see a concrete (hands-on) example and then understand how to do the problems more abstractly (without the chips).

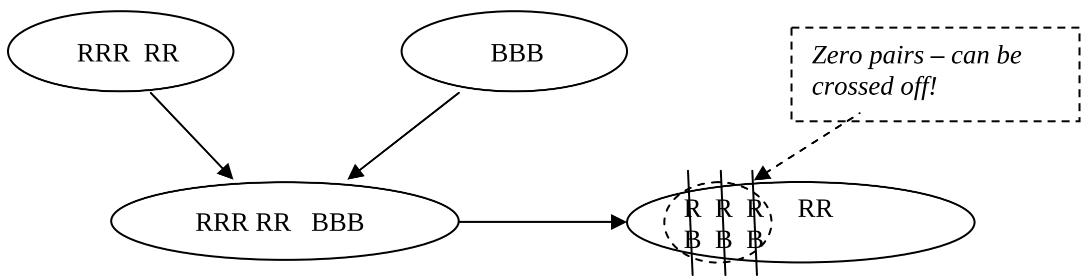
However, adding integers can be more challenging, and the chips are a nice way of working the problems when you get confused. Let's try a few of the same type of problems with integers to see how to work with this more challenging concept.

<u>More Examples:</u>

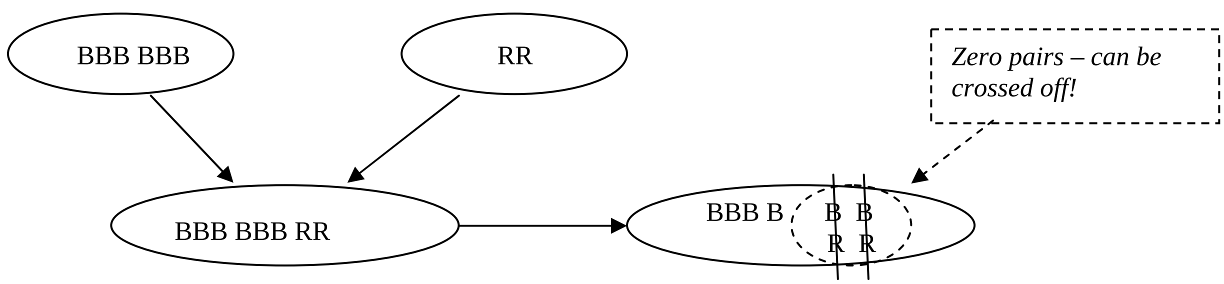
D) (– 5) + 3 E) 6 + (– 2)

Here's how to set up with chips:

D) First, write out the number – 5 (RRR RR) then the number 3 (BBB), and push the chip piles together – adding like terms. This shows that (– 5) + 3 = – 2, since we end with 2 red chips.




E) First, write out the number 6 (BBB BBB) then the number – 2 (RR), and push the chip piles together, and simplify with zero pairs because these are not like objects. This shows that $6+(-2)=4$, since we end with 4 black chips.




*In the last examples, the concept of zero pairs returns. There are 6 black chips and 2 red chips, but 2 black and 2 red make for two zero pairs; when you cross them off, you end up with only 4 black chips. We could rewrite this last portion by moving some of the red and black chips around to really see the zero pairs.*

**EXPLORE!** Draw a chip diagram and determine the value of the expression; use zero pairs when necessary. Make sure to show the operation of addition by pushing the two addends together.

A) ** (– 3) + 5

C) 4 + 2

B) (– 5) + (– 4)

D) 1 + (– 3)

# 3.5: Adding Integers Without Chips

Integers are the whole numbers and their opposites: …, – 3, – 2, – 1, 0, 1, 2, 3, 4, … There are negative integers (less than 0), positive integers (greater than 0), and then there is 0 itself. 0 is quite an incredible number as we saw with the addition properties. To examine how useful 0 is, let's explore some addition.

Interactive Exercise: Find some values that will make these equations true:

A) $7 + \_\_\_ = 0$ B) $-37 + \_\_\_ = 0$ C) $-3748 + \_\_\_\_\_\_\_ = 0$

Based on these problems, it seems that for each integer, there is another integer that we could add to make 0; this number is called the **additive inverse (or opposite)**. In general, this rule is called the **Additive Inverse Property**: $a + (-a) = (-a) + a = 0$ for all numbers *a*. 3 + (– 3) = 0, so the additive inverse of 3 is – 3. This is the property that zero pairs are based on.

Look at the definition: what is the additive inverse of 0? Explain why.

When we want to find the additive inverse of a number, we often write it like this: $-(\ )$. If we want the additive inverse of 5, we'd write $-(5)$. Since the additive inverse of 5 is negative 5, we could also write $-(5) = -5$.

The symbol – is used a lot in math, with often different meanings. Here's the current summary:

| Symbol | Meaning in words |
|---|---|
| $-5$ | Negative 5 |
| $-(5)$ | Opposite of five |
| $-(-5)$ | Opposite of negative five |

**EXPLORE!** Simplify these expressions and describe in words what it represents.

A) ** $-(-43)$

B) $-(35)$

C) $-(-940)$

D) $-(0)$

To add integers, we could combine the different types individually (negatives and positives). Let's practice that skill first. It links nicely to adding like terms as we did at the beginning of this unit.

Remember this: 5 negatives + 11 positives + 3 negatives + 8 negatives + 2 positives =

We can combine like objects – numbers that are negative will combine together, and numbers that are positive will combine together.

<u>Example:</u> $-3+7+(-4)+12+(-5)$

Here, look for all the negatives and positives. You could even write whether they are negative (–) or positive (+) under each number. Combine the like objects.

- There are 3 negatives, 4 negatives, and 5 negatives… which combine to be 12 negatives.
- There are 7 positives and 12 positives… which combine to be 19 positives.

We would write this as: $-3+7+(-4)+12+(-5)=19+(-12)$. Since these two objects are not like objects, at this point we'd have to be done. We will go further and simplify this completely later.

**<u>EXPLORE!</u>** Try a few on your own to combine the like objects quickly.

A) ** $-13+9+6+(-5)+2=$

C) $-31+87+(-24)+12+(-15)=$

B) $8+(-2)+7+(-6)+(-11)=$

D) $-63+19+(-21)+(-7)+(-3)=$

0 is the only number that is not positive and not negative, which again, makes it very special – it has no sign!

In mathematics, when we are discussing distance, we often use these symbols: $|\ \ |$. For the numbers that we experience most often, putting a number inside these vertical bars represents their distance to zero. $|13|$ is called the **absolute value** of 13, and represents the distance to zero… which is 13. This is why we would write $|13| = 13$.

**EXPLORE!** Find the following values and describe in words what it represents.

A) ** $|-43| =$

D) $-|-9| =$

B) $|23| =$

E) $-|14| =$

C) $-(-8) =$

F) $-|0| =$

In unit 2, we saw that integers have a sign and size. The concept of sign relates to positive, negative or zero. But the concept of size relates back to absolute value. If we want to find the absolute value of a number, the slang term for that is the **size**.

Sometimes in mathematics, the precise notation and terminology can be cumbersome, but is still important. The following table can help show the connection between the value of a number and the components (size and sign).

**<u>EXPLORE!</u>** Fill out the rest of the rows in the table; the first is done as an example.

| | **Value** | **Absolute value (Size)** | **Sign** |
|---|---|---|---|
| A) | 26 | 26 | Positive |
| B) ** | $-52$ | | |
| C) | | 15 | Positive |
| D) | 63 | | |
| E) | | 489 | Negative |
| F) | $-(-9)$ | | |
| G) | 0 | | |

This is key – changing the size of a number changes the value, and changing the sign of a number changes the value; the value doesn't exist without both of these parts. We could think of the value of a number being on a number line, with the sign indicating a direction and the size indicating a magnitude. We can use this to compare values directly.

**<u>EXPLORE!</u>** Compare the sizes of the following numbers by writing a symbol ($<$, $>$, or = ) in the box.

| | First Value | | Second Value |
|---|---|---|---|
| A) ** | $\|-42\|$ | | 17 |
| B) | $\|-42\|$ | | $-17$ |
| C) | $-2$ | | $-7$ |
| D) | $-2$ | | 7 |
| E) | $\|-2\|$ | | $\|7\|$ |
| F) | $\|-2\|$ | | $\|-7\|$ |
| G) | $\|-28\|$ | | $\|28\|$ |
| H) | $-\|-7\|$ | | $-(-7)$ |

Now we have the background to be able to add positive numbers and negative numbers – two objects that were unlike objects previously. First, it will be important to be able to determine the sign of the sum – whether the result is positive or negative (or 0) is extremely important to all future operations.

What is the value of $26+(-26)$? What property does this link to?

In the addition expression $26+(-37)$, which of the addends has a larger size (absolute value)?

In this case, 37 has a larger absolute value since $|26|<|-37|$. For the addition problem $26+(-26)$, the end result will be 0. So if we included more negatives to this, like in $26+(-37)$, there would still be the zero pair $26+(-26)$ with some extra negatives remaining. This is why the sum of $26+(-37)$ will be negative.

<u>Interactive Example:</u> What is the sign of $86+(-34)$ and why?

Use this new information to create a rule for addition of integers and the sign of the sum.

**<u>EXPLORE!</u>** Use your rule to quickly be able to identify the sign of the sum. Use the rule you created previously to help you circle the "sign" of the sum… positive (P), negative (N), or zero (Z).

| | | Sign |
|---|---|---|
| A) ** | $-58+78$ | P N Z |
| C) | $19+(-59)$ | P N Z |
| E) | $43+(-16)$ | P N Z |
| G) | $(-39)+(39)$ | P N Z |

| | | Sign |
|---|---|---|
| B) ** | $(-19)+(-32)$ | P N Z |
| D) | $(-43)+(26)$ | P N Z |
| F) | $(43)+(26)$ | P N Z |
| H) | $197+(-342)$ | P N Z |

Now we can put the concepts together and find the value of the sum (not just the sign).

- To add $-13+17$, we can rewrite the addend with the larger size in terms of the other addend.
- There are more positives than negatives, so we'll use the 17 and re-write it using a 13.
  - o 17 = 13 + ___
- Now we can rewrite the original expression using our properties of addition (can you name the properties used here?):

$$-13+17=-13+(13+4)=(-13+13)+4=0+4=4.$$

What if the numbers are larger? If you can't see a way to rewrite an addend, it really is okay. You can even break the problem into a few different pieces. Take something like $52+(-18)$. If you don't see how to rewrite 52 into an 18, then start with something close… like 20: $52+(-18)=32+20+(-18)$. This bring us much closer to the sum, since we now have 20, so we could rewrite the 20 as 2 + 18. The final result would look like this:

$$52+(-18)=32+20+(-18)=32+2+18+(-18)=32+2+0=34.$$

*Another method could be to do this a little different:*
$$52+(-18)=30+22+(-18)=30+4+18+(-18)=30+4+0=34.$$

Each way we do this it is going to be the same result, so it doesn't matter how you choose to regroup the addends!

**EXPLORE!** Find the value of the sums.

A) ** $(86)+(-47)=$

B) $19+(-59)=$

C) $(-19)+(-32)=$

D) $(-94)+(148)=$

E) 🖩 $197+(-342)=$

F) 🖩 $3{,}568+(-342)+139=$

**For Love of the Math:** *In higher level mathematics, vector quantities have a direction and magnitude. Each integer could be thought of this way, having a direction (sign) and magnitude (size). Being able to see a single number as both a position on the number line (value) and as the combination of sign and size is valuable.*

Now that integer addition is in our mathematical tool belt, there is another algorithm that helps complete addition very quickly. Since we know with zero pairs that $2+(-2)=0$, we could use that fact to quickly add numbers that are close to whole groups of tens or hundreds.

Create a rule based on this: If we want to find the sum more quickly by increasing the size of one addend, what do we need to do to the other addend?

**EXPLORE!** Rewrite these sums, but keep the same value. Then, once it is rewritten, quickly find the sum.

A) ** $29+76$

B) $89+135$

C) $3{,}129+897$

D) $4{,}599+1{,}377$

E) $593+797$

F) $43{,}999+5{,}204$

When working with addition and we're going to re-write addends, which of these methods will keep the same value?

| | If we… | (Example problem) | Then the value is… |
|---|---|---|---|
| A) ** | Add to both | $898+374 \Rightarrow 900+376$ | Same Different |
| B) | Subtract from both | $898+374 \Rightarrow 824+300$ | Same Different |
| C) | Add to one, subtract from the other | $898+374 \Rightarrow 900+372$ | Same Different |

**Concept Questions**

When dealing with integers, the sign of the sum depends on the value of the addends. The sum could be always positive (P), always negative (N), or sometimes positive and sometimes negative (S). Label each of the following expressions as P, S, or N. If the answer is P or N, explain why. But if the answer is S, give one example that shows a positive result and one example that shows a negative result.

| | **Expression** | **Sign (circle one)** | **Examples or Explanation** |
|---|---|---|---|
| D) | pos + pos | P S N | |
| E) | pos + neg | P S N | |
| F) | neg + pos | P S N | |
| G) | neg + neg | P S N | |

Use that information to help us grasp what happens with the following addends.

| | If B is… | Then A + B is… |
|---|---|---|
| H) ** | Positive | Greater than A Less than A Equal to A |
| I) | Negative | Greater than A Less than A Equal to A |
| J) | Zero | Greater than A Less than A Equal to A |

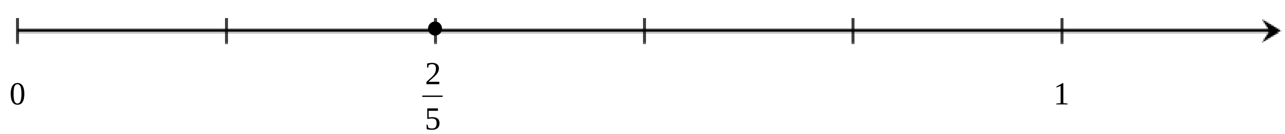
# 3.6: Adding Fractions (Rational Numbers)

Fractions are often referred to as rational numbers, and are written like ratios: $\frac{a}{b}$ where both *a* and *b* are integers, and $b \neq 0$. In a fraction, the number on top is called the **numerator** and the number on bottom is called the **denominator**. The word numerator comes from same idea as number, meaning to count; the word denominator comes from words meaning "classify." So when we see a fraction, we can interpret the numerator as "how many" and the denominator as "what type."

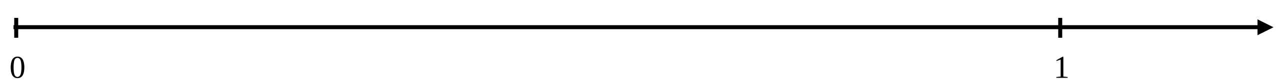
We could classify fractions on a number line, where the denominator is the number of pieces it takes to make one whole unit. So $\frac{1}{3}$ means that it takes 3 pieces to make a whole unit, and we have 1 of them. $\frac{2}{5}$ means that it takes 5 pieces to make a whole unit, and we have 2 of them.

Example: Plot the number $\frac{2}{5}$ on a number line.

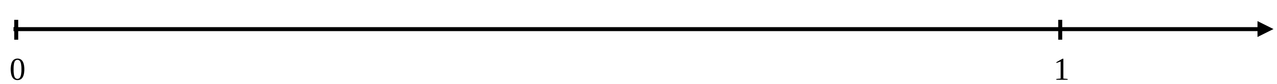
Since there are not enough parts to make one whole unit, start with a number line from 0 to 1. Sketch in four new tick marks so that the interval is split into five equal pieces. Starting at 0, count in two places and we have arrived at $\frac{2}{5}$.



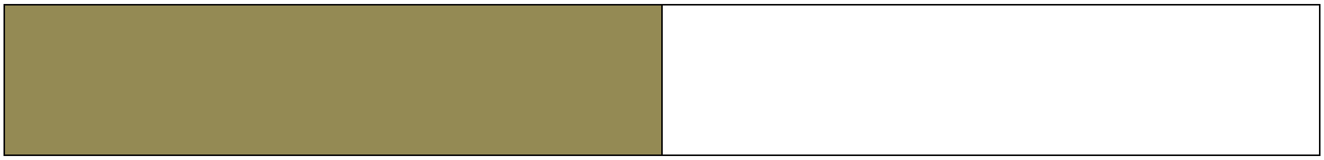

Interactive Example: Plot the number $\frac{3}{4}$ on a number line.



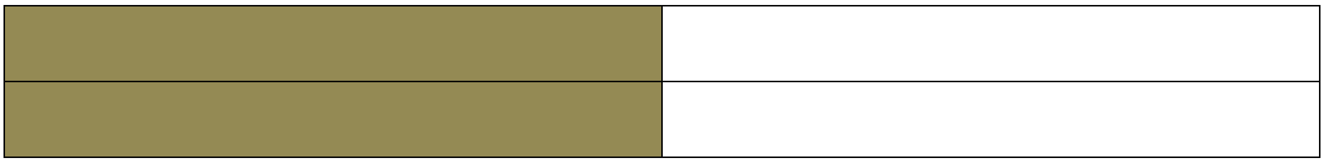

Interactive Example: Plot the number $\frac{2}{3}$ on a number line.



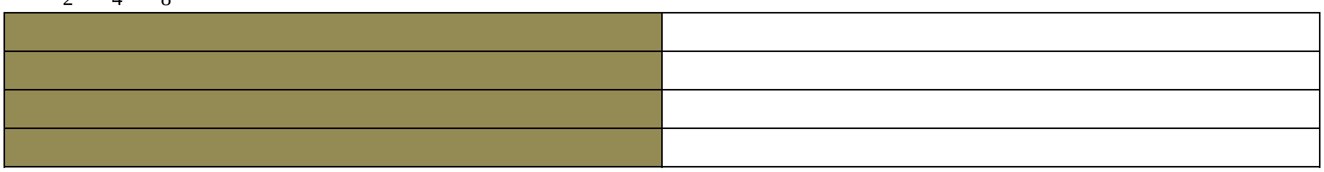
Remember in Unit 2 when we wrote numbers differently but kept the same value: 0374 = 374. Numbers are amazing because we can often rewrite them but not change the value. One great thing about fractions is the "Fundamental Law of Fractions" which allows us to do just that. Think about a candy bar shown below, and notice how half of the candy bar is shaded:



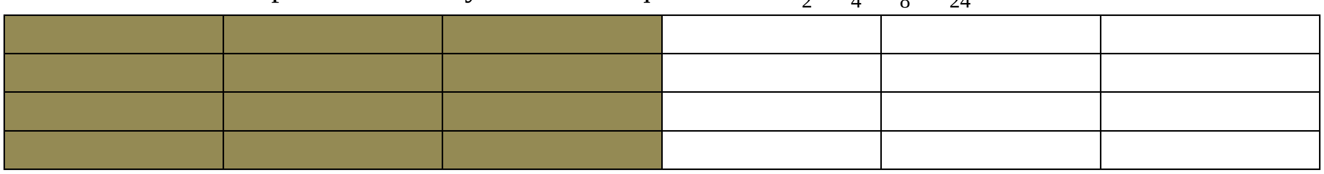
Now take the same candy bar, and cut it in half again… does it have the same amount of candy?



Visually, you can see that $\frac{1}{2} = \frac{2}{4}$. We could cut this in half again and we'd have equal amounts again, so $\frac{1}{2} = \frac{2}{4} = \frac{4}{8}$.



We could cut these pieces vertically into more equal slices: $\frac{1}{2} = \frac{2}{4} = \frac{4}{8} = \frac{12}{24}$



The **<u>Fundamental Law of Fractions</u>** (FLOF) says that $\frac{a}{b} = \frac{a \cdot n}{b \cdot n}$ for all non-zero values of *n*. So we say, "*if we don't like the way a fraction looks, we change the way it looks but keep the same value.*"

Find some equivalent fractions using FLOF: (find at least 3 equivalent fractions for each)

A) $\frac{2}{3} =$ ______ = ______ = ______

C) $\frac{1}{4} =$ ______ = ______ = ______

B) $\frac{5}{7} =$ ______ = ______ = ______

D) $\frac{7}{8} =$ ______ = ______ = ______

The fun part about fractions is that when we don't like the way they look, we just change them into something with the same *value* but a different look! This helps us to be able to determine relative sizes of fractions so we can see which is bigger.

Example: Which of these is larger: $\frac{3}{4}$ or $\frac{7}{8}$?

We can use FLOF to rewrite $\frac{3}{4}$ to have the same denominator: $\frac{3}{4}=\frac{3\times 2}{4\times 2}=\frac{6}{8}$. Now the question is much easier… which of these is larger: $\frac{6}{8}$ or $\frac{7}{8}$?

Example #2: Which of these is larger: $\frac{3}{5}$ or $\frac{5}{8}$?

We can use FLOF to rewrite $\frac{3}{5}$ and $\frac{5}{8}$ to have the same denominator – we'll use each denominator as the factor to FLOF with: $\frac{3}{5}=\frac{3\times 8}{5\times 8}=\frac{24}{40}$ and $\frac{5}{8}=\frac{5\times 5}{8\times 5}=\frac{25}{40}$. Now the question is much easier… which of these is larger: $\frac{24}{40}$ or $\frac{25}{40}$?

**EXPLORE!** Compare the sizes of the following numbers by writing a symbol ( $<$, $>$, or = ) in the box.

| | First Value | | Second Value |
|---|---|---|---|
| A) ** | $\frac{2}{7}$ | | $\frac{3}{8}$ |
| B) | $\frac{9}{16}$ | | $-\frac{5}{8}$ |
| C) | $-\frac{5}{8}$ | | $-\frac{8}{13}$ |
| D) | $\frac{5}{20}$ | | $\frac{1}{4}$ |
| E) 🖩 | $\frac{29}{43}$ | | $\frac{48}{71}$ |

When we added things that were like objects, it was pretty quick when they were like objects, and adding fractions is just like this.

<u>Interactive Examples:</u> Remember like objects… can we add the following quickly?

A) 3 sevenths + 2 sevenths + 4 sevenths =

B) 3 fourths + 1 half + 1 half + 3 fourths + 3 halves =

<u>Examples:</u> We could represent the previous questions in fraction notation:

A) $\frac{3}{7}+\frac{2}{7}+\frac{4}{7}=\frac{3+2+4}{7}=\frac{9}{7}$

B) $\frac{3}{4}+\frac{1}{2}+\frac{1}{2}+\frac{3}{4}+\frac{3}{2}= \frac{6}{4}+\frac{5}{2}$

From the first page, the denominator represents the size or type of fraction (what type), while the numerator represents the quantity (how many). That's why when adding fractions with the same denominator, we just add the numerators – we can only add like objects, and in this case, the like object is the denominator.

**<u>EXPLORE!</u>** Add fractions, the first has been done for you.

A) $\frac{3}{9}+\frac{2}{9}+\frac{5}{9}=\frac{3+2+5}{9}=\frac{10}{9}$

C) ** $-\frac{2}{19}+\frac{5}{19}+\frac{8}{19}=$

B) $\frac{3}{14}+\frac{6}{14}=$

D) $\frac{31}{184}+\left(-\frac{17}{184}\right)=$

Create a rule for how to find the sum when the denominators are the same: $\frac{a}{b}+\frac{c}{b}=$

But how could we add $\frac{2}{3}+\frac{1}{2}$? The denominators are not the same, so we will use FLOF to make them the same, called **<u>common denominators</u>**. We multiply each fraction by a number to get us to common denominators. These numbers are often called "common multiples" and this is a way for us to keep the *same value*, but have the *different look* that makes our addition much easier!

$$\underbrace{\frac{2}{3}+\frac{1}{2}}_{\text{Not the Same}} = \underbrace{\frac{2\cdot 2}{3\cdot 2}+\frac{1\cdot 3}{2\cdot 3}}_{\text{Using FLOF to get common denominators}} = \underbrace{\frac{4}{6}+\frac{3}{6}}_{\text{Same Denominators}} = \frac{7}{6}$$

**EXPLORE!** Try a few on your own.

A) ** $\frac{3}{4}+\frac{1}{8}$

B) ** $\frac{7}{12}+\frac{5}{8}$

C) $\frac{3}{2}+\frac{5}{8}$

D) $\left(-\frac{8}{9}\right)+\left(-\frac{2}{3}\right)$

E) $\frac{7}{20}+\left(-\frac{11}{30}\right)$

F) $\frac{3}{5}+\frac{7}{15}$

G) $\left(-\frac{3}{10}\right)+\frac{7}{15}$

H) $\frac{8}{11}+\frac{3}{4}$

I) 🖩 $\frac{7}{31}+\frac{6}{18}$

J) 🖩 $\frac{11}{49}+\left(-\frac{31}{42}\right)$

Create a rule for how to find sums when denominators aren't the same: $\frac{a}{b}+\frac{c}{d}=$

# 3.7: Adding Decimals

Decimals are extremely quick to add together, as long as you remember place values!

In order to add 2.04 and 3.9, remember that we need to add like objects together. The "ones" should be added, the "tenths" should be added, etc. Perhaps the best way to accomplish this is by lining up the decimal points in order to keep the place values the same.

2.04 + 3.9 would be re-written vertically as: $\begin{array}{r} 2.04 \\ \underline{+3.9\phantom{0}} \end{array}$. It may also be a good idea to put extra 0's after the last decimal place; this way, all the digits are lined up in place value notation. Rewriting 3.9 = 3.90 is another way to show: "*if we don't like the way a number looks, we change the way it looks but keep the same value.*"

$$\begin{array}{r} 2.04 \\ \underline{+3.90} \\ 5.94 \end{array}$$

**EXPLORE!** Try a few on your own using this method; write additional 0's where needed but don't change the value of the addends.

A) ** $14.5+0.903+0.0247$

B) 🖩 $16.45+3.718$

C) $216.5+7.73$

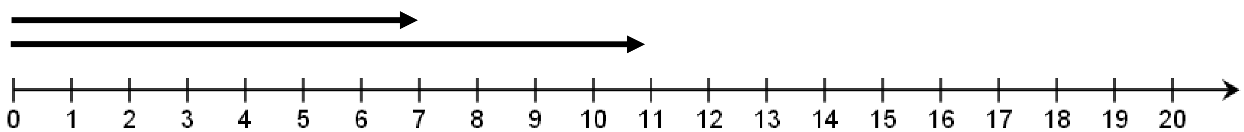
D) $6.05+2.803+19.2$

# 3.8: Adding Values on a Number Line

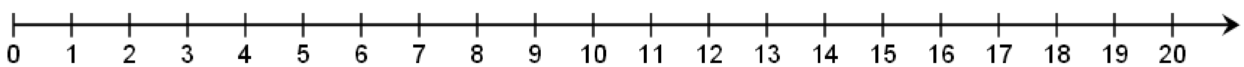
Number line addition can be quick and fun. It allows for us to get an idea of where a number will be located quickly, but we would need a bit more work to get the actual value.

On a number line, we could always create an arrow from 0 to the location of our number or numbers.

Examples: Represent 7 and 11 on the number line.
The number line below shows the arrows that would represent the numbers 7 and 11.

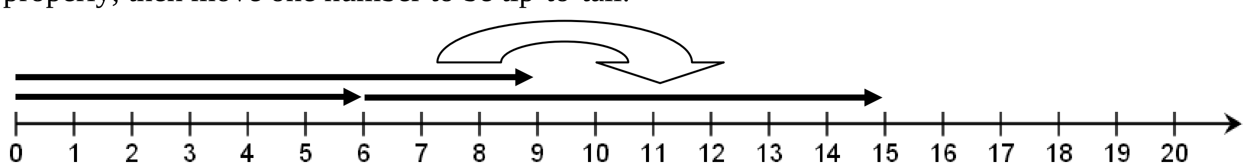



Interactive Examples: Represent the three separate numbers 12, 6, and 9 on the number line.



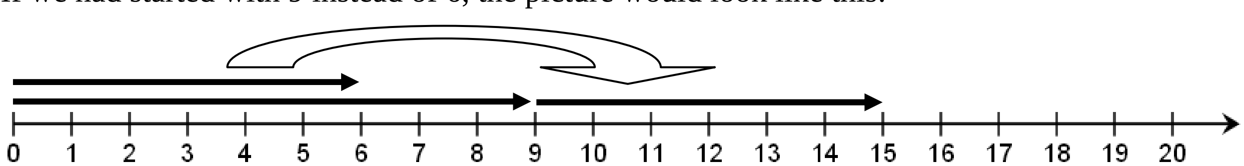

In order to add numbers on a number line, we create the arrows, and then "adding" them is putting them **tip-to-tail**. That means we start the next arrow where the first one ended.

Examples: Represent 6 + 9 and 9 + 6 on the number line.
To represent the sum of 6 and 9 on a number line, we would draw the arrows and the position them properly, then move one number to be tip-to-tail.




This picture shows that 6 + 9 = 15.

If we had started with 9 instead of 6, the picture would look like this:

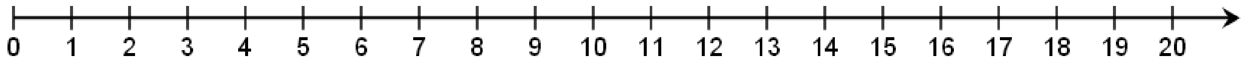



This second picture shows 9 + 6 = 15.

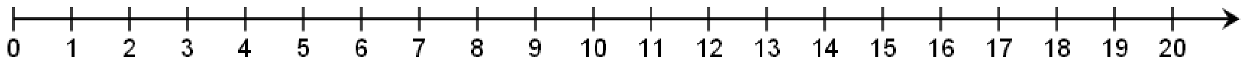
What you can see visually is that the order of addition doesn't matter and that the same sum would result with either direction. 6 + 9 = 9 + 6. This is a visual indication of which property?

**EXPLORE!** Try a few on your own. Be sure to draw the arrows.

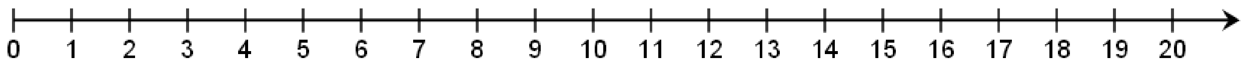
A) 4 + 9




B) 7 + 12

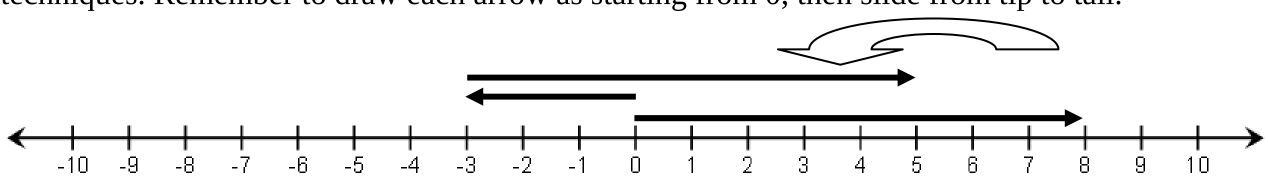



C) 2 + 5 + 7




To this point, the only examples have involved positive numbers; but we could use these tables with lots of values – positive, negative, fractions, and even complicated looking numbers like square roots.

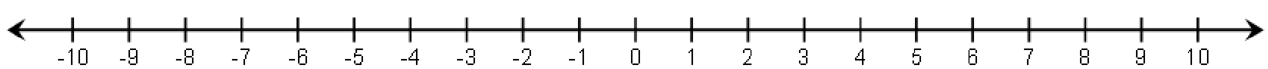
<u>Examples:</u> Let's try one with negative numbers. Find the value of $(-3)+8$ using the number line techniques. Remember to draw each arrow as starting from 0, then slide from tip to tail.



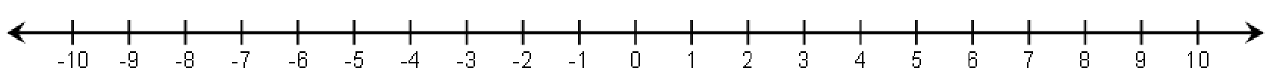

Notice how the resulting arrow ends at positive 5. That shows the result: $(-3)+8=5$.

**EXPLORE!**

A) Find the value of $(-3)+(-5)$ using the number line techniques.



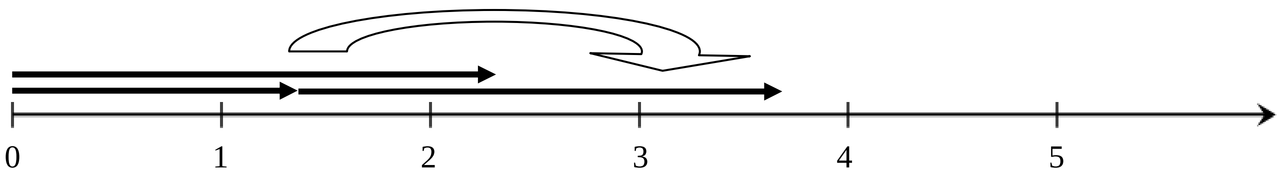

B) Find the value of $(-3)+6+(-7)$ using the number line techniques.



Sometimes in math, the exact answer is what we are looking for; other times, a quick approximation would be ideal. This approximation idea is another great way that number lines can help us!

If we wanted an approximation for $\sqrt{5}+\sqrt{2}$, we could use number lines! Draw an arrow for each and put them tip-to-tail. From Unit 2 in this class, we know that $\sqrt{5}$ is between 2 and 3, and is closer to 2. Also, $\sqrt{2}$ is between 1 and 2, and is a bit closer to 1. That’s all we need for the arrows!

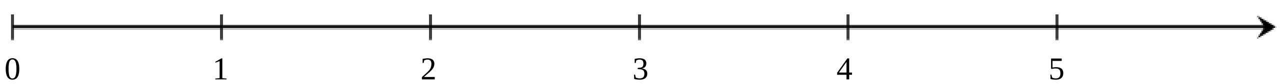
<u>Example:</u> Represent $\sqrt{5}+\sqrt{2}$ on a number line.



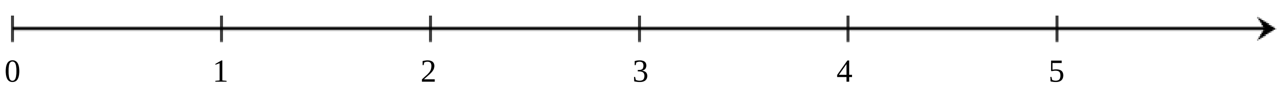

From the picture we can see that $\sqrt{5}+\sqrt{2}$ is between 3 and 4, with a value of about $3\frac{3}{4}$. Check this result using your calculator: $\sqrt{5}+\sqrt{2} \approx$ _____________. How close were we?

**<u>EXPLORE!</u>** Approximate the sum using the number line:

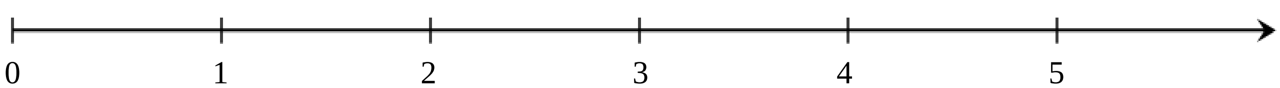
A) $\sqrt{4}+\sqrt{9}$




B) $\sqrt{2}+\sqrt{7}$




C) $\sqrt{6}+\sqrt{10}$




Based on the last two examples, is it true that $\sqrt{2}+\sqrt{7}=\sqrt{9}$? What about $\sqrt{6}+\sqrt{10}=\sqrt{16}$? Can we add the square roots like this?

Lastly, we can also do number lines with values that we don't know. Let's use the arrow number line techniques on the following problems. Because we want you focusing on the big picture, we'll split the number line into regions. We'll pick the region that best fits our number.

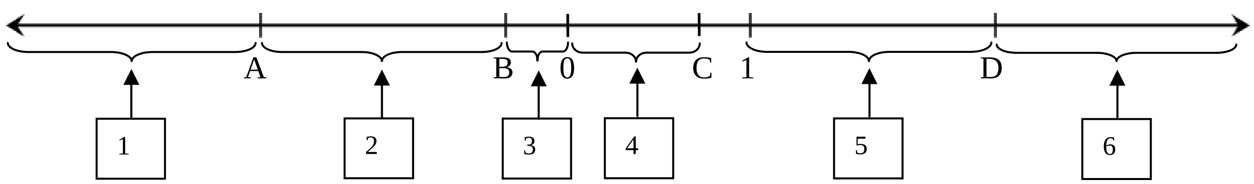
Remember from a previous section:

- If B is positive, then A + B is greater than A.
- If B is negative, then A + B is less than A.
- If B is zero, then A + B is equal to A.

To do this, we'll need to create our arrows and then tip-to-tail, even though we don't know the value of the variables, we can still represent the sum!

Here is our standard number line for these problems.


Then we put in the regions for our answers:




Let's practice a few parts with the first type.

**EXPLORE!** Find the values on the number line:

A) ** – A

C) – B

E) – C

B) – 1

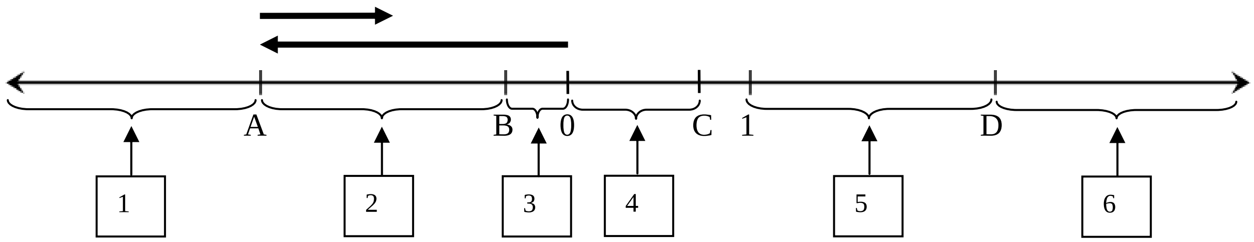
D) 2

F) – 2


Once you can find number locations, we can move into operations.

Example: Find the value of A + C.

Now we'll try out the problem of A + C. Again, tip to tail show that we will end up in region 2.




**EXPLORE!** Try a few on your own.

A) ** Find the region that best approximates B + D.

B) ** Find the region that best approximates C + (– B).

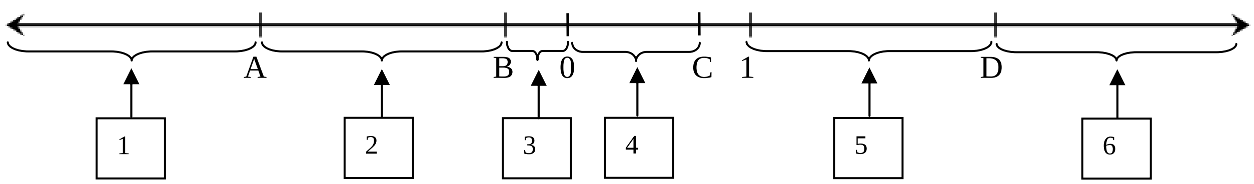
C) Find the region that best approximates B + C.

D) Find the region that best approximates 1 + C.

E) Find the region that best approximates (– 1) + B.

F) Find the region that best approximates D + (– 1).

G) Find the region that best approximates C + (– B).



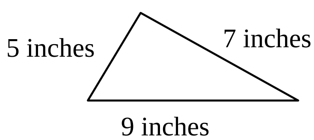
# 3.9: Applications of Addition and Concept Questions

Geometric applications:
If you wanted to find the distance around a geometric shape, it is called **perimeter**. To find the perimeter of an object, like the rectangle below, we add up all the lengths of the sides.

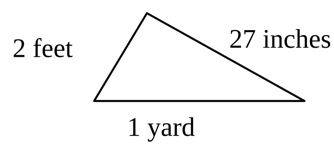
A) Find the perimeter of the rectangle:


B) Find the perimeter of the rectangle:

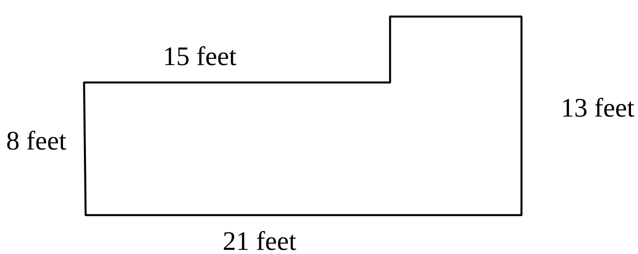

C) Find the perimeter of the triangle:




D) Find the perimeter of the triangle:




E) Find the perimeter of the shape below:




F) What formula could we use for the perimeter of a rectangle?

Sport applications:

A) A diver is at a depth of 35 feet (below the water) and rises 19 feet. What is the diver's new depth?

Money applications:

B) In a checking account, there is $451. The account owner puts in $12 and then the bank adds a fee of $34. What is the ending balance?

Algebraic applications:

In algebra, we use addition to combine like objects… but often the objects are variables. An example would be to add the following expression: $7x+3y+5x+(-2y)$. Once we examine the expression and find the common terms, we can add the like objects quickly: $7x+3y+5x+(-2y)=12x+y$.

C) ** $-13x+(-7y)+9x+12y$

D) $13x+7x^2+9x+(-2x^2)$

E) $13x+7y+9+(-8x)$

F) $-9x+14y+(-7x)+(-32y)$

# 3.10: Addition Summary

When dealing with addition, the key is to add like objects. Here are some wrap up questions to tie all the concepts together.

Explain/Summarize the following addition properties:

- Commutative Property of Addition

- Associative Property of Addition

- Additive Identity Property

- Additive Inverse Property

**Concept Questions**

When dealing with integers, the sign of the sum depends on the value of the addends. The sum could be always positive (P), always negative (N), or sometimes positive and sometimes negative (S). Label each of the following expressions as P, S, or N. If the answer is P or N, explain why. But if the answer is S, give one example that shows a positive result and one example that shows a negative result.

| | **Expression** | **Sign (circle one)** | **Examples or Explanation** |
|---|---|---|---|
| A) | pos + pos | P S N | |
| B) | pos + neg | P S N | |
| C) | neg + pos | P S N | |
| D) | neg + neg | P S N | |

| | If B is… | Then A + B is… | | |
|---|---|---|---|---|
| E) ** | Positive | Greater than A | Less than A | Equal to A |
| F) | Negative | Greater than A | Less than A | Equal to A |
| G) | Zero | Greater than A | Less than A | Equal to A |

# <u>3.11: Subtraction</u>

Subtraction is the inverse operation of addition, but for many people, subtraction is a not really much different than addition.

**<u>For Love of the Math:</u>** *In higher level mathematics where the number sets are examined in higher detail, there are words describing the real numbers as a* ***<u>field</u>*** *– which is a set of numbers and two operations that satisfy certain properties. For the real numbers, those two operations are addition and multiplication. Over time, it became simpler to create new operations that could be tied back to the field properties. As we will see, subtraction is really a way to describe a type of addition – and division is a way to describe a type of multiplication.*

It's really helpful to be able to define subtraction so that we're clear on what we mean by $a - b = c$.

**<u>Definition</u>**: The subtraction of $b$ from $a$ is written $a - b$: $a - b = c$ will be true if, and only if, $c + b = a$ and the sum, $c$, must exist and be unique. From the equation $a - b = c$, $a$ is called the **<u>minuend</u>**, $b$ is called the **<u>subtrahend</u>**, and $c$ is called the **<u>difference</u>**.

**<u>NOTE:</u>** *The terminology used here for the minued and subtrahend might seem complicated. Often, subtraction is done without knowledge of these terms. In fact, if you don't know these terms, you'd still be able to perform the operation. We hope you won't get too hung up on the names here as we won't use these two terms often. The most important part of this discussion is that there are specific names for both the first and second numbers in subtraction – something not true with addition.*

Processes exist to make the difference easier to compute. Perhaps in your life, you've seen methods of doing subtraction, called algorithms. Here, we'll show a few algorithms for subtraction of whole numbers – which will guide all of the other subtractions we see.

**<u>Algorithm #1:</u>** Missing Addend or Cashier's technique.

In this algorithm, we try to think of the value needed to be added to get back to the minuend. As an example, if we were asked to find 10 – 7, we might think about what is needed to add to 7 to get 10. Since we know 3 + 7 = 10, then 10 – 7 = 3. This algorithm is often used when the values are small and easy to compute mentally. If numbers get larger, then we just count up from the smaller number until we get to the larger number.

<u>Examples:</u>

A) 14 – 10 =　　　B) 200 – 194 =　　　C) 341 – 275 =

<u>Solutions:</u>

A) With 14 – 10 = $x$, we could rewrite this with the definition as $x + 10 = 14$. What needs to be added to 10 to end with 14… it's 4! So 14 – 10 = 4.

B) 200 – 194 = $x$, where $x + 194 = 200$. Here we need to add up to 200, so we need just 6 more. This is why 200 – 194 = 6.

C) 341 – 275 = $x$, where $x$ + 275 = 341. This one is much tougher, right? It's pretty hard to see what you need to add to 275 to end with 341. We could add to 275 and see what we come up with…

i. Start with 275. *Since we need to end with a 1 in the ones place, what should we add to 275? We need to add 6.*
ii. 275 + 6 = 281. *Now we need to get up to 341, so perhaps get up to 300 first by adding 20.*
iii. 281 + 20 = 301. *Now we need to get up to 341, so add an additional 40.*
iv. 301 + 40 = 341. *We made it! By figuring out how much was added, we'll find the difference that we're looking for*.
v. Notice we added 6 and 20 and 40, or 6 + 20 + 40 = 26 + 40 = 66.
vi. Since 275 + 66 = 341, we can say that 341 – 275 = 66.

The reason this technique is often called the cashier's method is that when you get your change back, you could count the change back this way to get the total. And you could even try to be a cashier and see how to count up to the total with $20, $10, $5, and $1.

Cashier's method with verbal description for $110 – $77; this would come from someone paying $110 for an item that costs $77.

| **Cashier hands customer…** | **Cashier says this to count up** |
|---|---|
| The item purchased | $77 |
| $20 | $97 |
| $10 | $107 |
| $1 | $108 |
| $1 | $109 |
| $1 | $110 |

Adding up all the money handed back gives the difference: 20 + 10 + 1 + 1 +1 = 33. So we could write the subtraction as 110 – 77 = 33.

**EXPLORE!** Try a few using this algorithm:

A) 92 – 36 =

B) 613 – 364 =

**<u>Algorithm #2:</u>** Partial Differences.

In this algorithm, we subtract numbers quickly, and remember that if we take away more than we start with, the result must be negative. Starting with 5 and subtracting 7 would leave a result of – 2. We would write this as 5 – 7 = – 2. Just like with partial sums, we'll do this as one part at a time by place value; when finished, we'll add up the results! Here's the process for 535 – 237:

*Step 1:* Write the problem and line up the place values and draw a line underneath.

*Step 2:* Starting with the ones place value, subtract the digits and write the result in the ones place value under the line – making sure to write the correct sign. 5 – 7 = – 2.

*Step 3:* Move to the next place value (tens), subtract the digits and write the result in the tens place value under the line – making sure to write the correct sign. 3 – 3 = 0.

*Step 4:* Move to the next place value (hundreds), subtract the digits and write the result in the hundreds place value under the line – making sure to write the correct sign. 5 – 2 = 3.

*Step 5:* When finished, draw a line under the last partial difference and then add these partial differences and the result is the difference sum of the original problem: $535 - 237 = 298$.

| ***Step 1*** | ***Step 2*** | ***Step 3*** | ***Step 4*** | ***Step 5*** |
|---|---|---|---|---|
| 535<br>– 237 | 535<br>– 237<br>– 2 | 535<br>– 237<br>– 2<br>00 | 535<br>– 237<br>– 2<br>00<br>300 | 535<br>– 237<br>– 2<br>00<br>300<br>298 |

<u>Interactive Exercises:</u>

A)

$$\begin{array}{r} 92 \\ \underline{-\ 64} \end{array}$$

B)

$$\begin{array}{r} 613 \\ \underline{-\ 364} \end{array}$$

**<u>NOTE:</u>** This technique is quick, but since there are both positive and negative numbers in the partial differences, it is often not taught.

**Algorithm #3:** Traditional Algorithm

In this algorithm, we re-group numbers when necessary and line them up with place values vertically. The key to this algorithm is the ability to rewrite numbers in different forms. Remember in the addition sections where we found many ways to re-write 92, like 92 = 82 + 10. For a problem like $92-36$, we could think of many way to rewrite the 92 and 36 in order to make this easier.

92 could be written as 92 = 80 + 12 and 36 could be written as 36 = 30 + 6. When subtracting, we'll subtract like objects (tens and ones): 80 – 30 = 50 and 12 – 6 = 6. The difference is 50 + 6 = 56. So $92-36=56$.

However, when we write this by hand, we often cross out a number and write other numbers. Traditionally, this was called borrowing. However, we no longer call it borrowing since you're not really borrowing anything. Instead, you'll hear us refer to this as "regrouping," "decomposing," or "fair-trading." Here's the process for 535 – 237:

*Step 1:* Write the problem vertically, line up the place values, and draw a line underneath.

*Step 2:* Starting with the ones place value, subtract quickly if there are more ones in the minuend than the subtrahend. In this case, there are 7 ones to subtract from 5 ones. Having more ones would make this easier, so we decompose one group of ten from the 535 which turns 1 ten into 10 ones. Scratch through the 3 groups of ten and write a 2 above it (indicating the number of tens remaining), and then put 10 more ones with the 5 ones already there… now a total of 15. Then subtract 15 – 7 to get 8. Record the result as the number of ones, and write 8 under the line in the ones place value.

*Step 3:* Move to the next place value (tens), and subtract quickly if possible. There are 2 tens left and we're trying to subtract 4 tens. Once again, having more would make this easier. We decompose one group of hundred from the 535 which turns 1 hundred into 10 tens. Scratch through the 5 groups of hundred and write a 4 above it (indicating the number of hundreds remaining), and then put 10 more tens with the 2 tens already there… now a total of 12. Then subtract 12 – 3 to get 9. Record the result as the number of tens, and write 9 under the line in the tens place value.

*Step 4:* Move to the next place value (hundreds), and subtract quickly. In this case, we can do this quickly since 4 hundreds – 2 hundreds = 2 hundreds. Write this result in the hundreds place under the line which shows 535 – 237 = 298.

| ***Step 1*** | ***Step 2*** | ***Step 3*** | ***Step 4*** |
|---|---|---|---|
| $\begin{array}{r} 535 \\ \underline{-237} \end{array}$ | $\begin{array}{r} \scriptstyle 2\,15 \\ 5\,\not{3}\,\not{5} \\ \underline{-2\,3\,7} \\ 8 \end{array}$ | $\begin{array}{r} \scriptstyle 4\,12\,15 \\ \not{5}\,\not{3}\,\not{5} \\ \underline{-2\,3\,7} \\ 9\,8 \end{array}$ | $\begin{array}{r} \scriptstyle 4\,12\,15 \\ \not{5}\,\not{3}\,\not{5} \\ \underline{-2\,3\,7} \\ 2\,9\,8 \end{array}$ |

When using this algorithm, we typically write the larger whole number as the minuend (top number) and the smaller whole number as the subtrahend (bottom number).

**Algorithm #4:** Subtract from the Base

This is a modification of the “Traditional” algorithm but makes it a bit easier in the re-grouping phase… as it is always a 10! This algorithm came from thinking about a few lists of subtractions that you could do in your head. Look at these two lists and think about which list you’d rather work with.

| List A | | | |
|---|---|---|---|
| 12 – 8 | 18 – 9 | 15 – 7 | 14 – 6 |

| List B | | | |
|---|---|---|---|
| 10 – 8 | 10 – 9 | 10 – 7 | 10 – 6 |

Most people would pick List B, since these subtraction problems are easier to work with. Much of this algorithm looks like the traditional algorithm, but each time you regroup, just regroup with a 10 every time. We’ll use 535 – 237 again.

*Step 1:* Write the problem vertically, line up the place values, and draw a line underneath.

*Step 2:* Starting with the ones place value, subtract quickly if there ones are more in the minuend than the subtrahend. Just like in the traditional case, having more ones would make this easier, so we decompose one group of ten from the 535 which turns 1 ten into 10 ones. Scratch through the 3 groups of ten and write a 2 above it (indicating the number of tens remaining), and then put 10 more ones with the 5 ones already there, but leave it written separately as 10 and 5. Then subtract 10 – 7 to get 3 and combine with the other 5 ones, leaving 8. Record the result as the number of ones, and write 8 under the line in the ones place value.

*Step 3:* Move to the next place value (tens), and subtract quickly if possible. There are 2 tens left and we’re trying to subtract 4 tens. Once again, having more would make this easier. We decompose one group of hundred from the 535 which turns 1 hundred into 10 tens. Scratch through the 5 groups of hundred and write a 4 above it (indicating the number of hundreds remaining), and then put 10 more tens with the 2 tens already there, but leave it written separately as 10 and 2. Subtract 10 – 3 to get 7, and combine with the other 2 tens leaving 9 tens. Record the result as the number of tens, and write 9 under the line in the tens place value.

*Step 4:* Move to the next place value (hundreds), and subtract quickly. In this case, we can do this quickly since 4 hundreds – 2 hundreds = 2 hundreds. Write this result in the hundreds place under the line which shows 535 – 237 = 298.

| ***Step 1*** | ***Step 2*** | ***Step 3*** | ***Step 4*** |
|---|---|---|---|
| 535<br>– 237 | 2 10<br>5~~3~~5<br>– 237<br>8 | 10<br>4 2 10<br>~~5~~ ~~3~~ 5<br>– 2 3 7<br>9 8 | 10<br>4 2 10<br>~~5~~ ~~3~~ 5<br>– 2 3 7<br>2 9 8 |
| | Think 10 – 7 = 3<br>3 + 5 (other ones) = 8 | Think 10 – 3 = 7<br>7 + 2 (other tens) = 9 | |

**EXPLORE!**

A) **

715<br>– 359

B)

923<br>– 157

**Algorithm #5:** Equal Addends

This final algorithm is used sometimes when there would be a lot of regrouping. Perhaps some of you remember problems like 2000 – 543, where there is so much regrouping, that it is easy to make a mistake. This algorithm is very nice for those types of problems, and is based on a nice fact about subtraction.

19 – 6 = 13. 20 – 7 = 13. 21 – 8 = 13. 22 – 9 = 13. 23 – 10 = 13.

Notice how adding 1 to both minuend and subtrahend will result in the same difference. Taking one away from both would also result in the same difference. Subtraction measures how far apart two numbers are. So to find 2,000 – 543, we could just subtract 1 from each number. It creates a new problem that is easier to work with but results in the same difference (distance between) as the original.

$$\begin{array}{r} 2000 \\ -\underline{543} \end{array} \text{ (starting point)} \rightarrow \begin{array}{r} 1999 \\ -\underline{542} \end{array} \text{ (subtracting 1 from each number)} \rightarrow \begin{array}{r} 1999 \\ -\underline{542} \\ 1457 \end{array} \text{ (subtract)}$$

**EXPLORE!** Rewrite these so they would be easier to subtract (but keep the same value).

A) ** 5,000 – 185 =

B) 748 – 399 =

C) 813 – 297 =

D) ** 6,004 – 5,198 =

E) 8,001 – 499 =

F) (do differently ) 8,001 – 499 =

**EXPLORE!** Using what you did above, find the difference .

A) ** 5,000 – 185 =

B) 748 – 399 =

C) 813 – 297 =

D) ** 6,004 – 5,198 =

E) 8,001 – 499 =

F) (do differently ) 8,001 – 499 =

When working with subtraction, which of these methods will keep the same value?

| | If we… | (Example problem) | Then the value is… |
|---|---|---|---|
| A) ** | Add to both | $898 - 374 \Rightarrow 900 - 376$ | Same Different |
| B) | Subtract from both | $902 - 374 \Rightarrow 899 - 371$ | Same Different |
| C) | Add to one, subtract from the other | $898 - 374 \Rightarrow 900 - 372$ | Same Different |

With subtraction defined in terms of addition, it's still important to remember that the objects must be like terms. What we just saw with different algorithms is that there are different ways to compute the solution to a subtraction problem – you may change your algorithm based on the problem you encounter. Try some problems here and see if changing your algorithms helps make it easier to find the results!

**EXPLORE!** Which algorithm would you use to compute these differences?

| | Difference | Partial | Traditional | Subtract From Base | Equal Addends |
|---|---|---|---|---|---|
| A) ** | 619 – 347 | | | | |
| B) | 2,300 – 198 | | | | |
| C) | 3,524 – 1,203 | | | | |
| D) | 984 – 658 | | | | |
| E) | 4,701 – 563 | | | | |
| F) | 4,000,000 – 265,843 | | | | |

**EXPLORE!** Use your algorithm choice to compute the differences below.

A) 619 – 347 =

D) 984 – 658 =

B) 2,300 – 198 =

E) 4,701 – 563 =

C) 3,524 – 1,203 =

F) 4,000,000 – 265,843 =

G) 🖩 138,302,594 – 11,987,452 =

# 3.12: Subtracting Integers using Chips

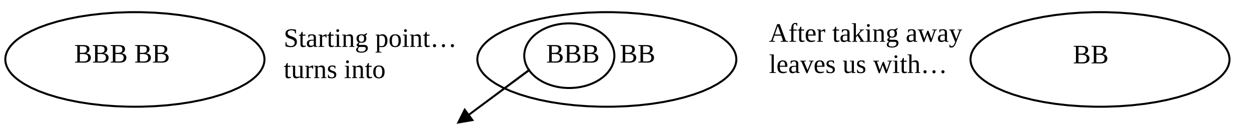
As before, we'll use "B" for black chips to represent positive integers and "R" for red chips to represent negative integers. The ideas from before are still important:

i. Zero pairs are still critical – the idea that BR is worth 0.
ii. Rewriting numbers in different representations:
    a. 5 can be represented as BBBBB or BBBBB BR or BBBBB BBRR.

Remember that addition was "adding to" something, and that the inverse operation would be "taking away." Using the chips, we will see how taking away from a number gives the result of subtraction.

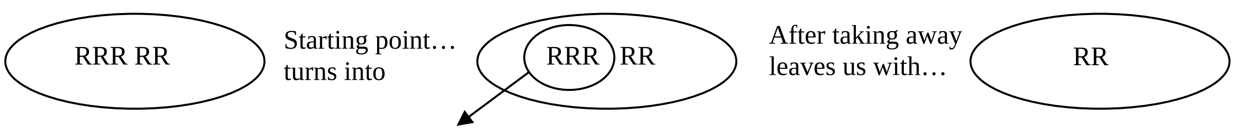
To show our first example with chips, we will start slowly: 5 – 3.

First, write out the number 5 (BBB BB) then take away the value 3 (BBB). Whatever remains is the difference, and in this case, 2 black chips are left: 5 – 3 = 2. Here's the picture that you would draw.



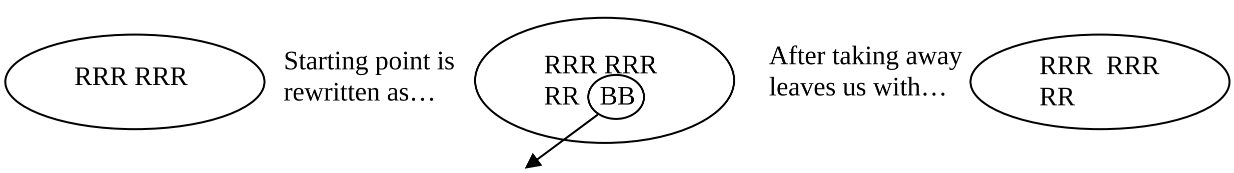

Now we'll try another one that seems much more challenging: (– 5) – (– 3).

First, write out the number – 5 (RRR RR) then take away the value – 3 (RRR). Whatever remains is the difference, and in this case, 2 red chips are left: (– 5) – (– 3) = – 2. Here's the picture that you would draw.




Sometimes, there isn't enough of one type to take away – which is where the ability to represent a value differently comes in handy. Let's try this one: (– 6) – 2.

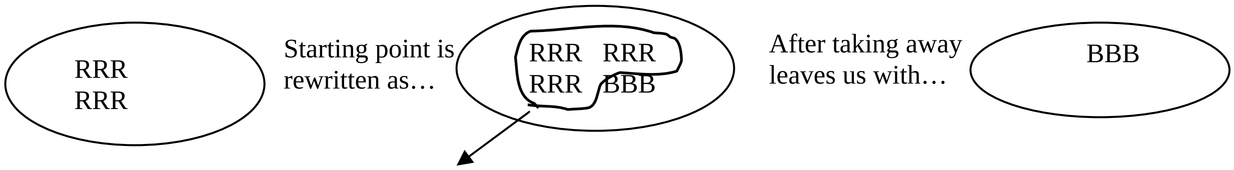
First, write out the number – 6 (RRR RRR) then take away the value 2 (BB). But there's a problem because there are no black chips in the first number. Guess what – we can rewrite the number – 6 by using zero pairs. Once finished, we'll take away our 2 black chips and we are left with 8 red chips. Whatever remains is the difference, and in this case, (– 6) – 2 = – 8. Here's the picture that you would draw.




It is pretty clear that we'll need to be thinking about what we're taking away to use this technique, and if we don't have what we want, then consider putting in zero pairs that will allow the subtraction.

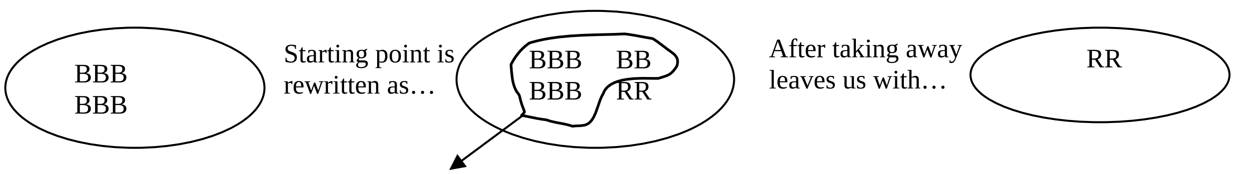
Now we'll work another example that illustrates the same concept: (– 6) – (– 9).

First, write out the number – 6 (RRR RRR) then take away the value – 9 (RRR RRR RRR). But there's a problem because there are not enough red chips in the first number. Guess what – we can rewrite the number – 6 by using zero pairs. Once finished, we'll take away our 9 red chips. Whatever remains is the difference, and in this case, (– 6) – (– 9) = 3. Here's the picture that you would draw.




Now we'll work one last example that illustrates the same concept: 6 – 8.

First, write out the number 6 (BBB BBB) then take away the value (BBB BBB BB). But there's a problem because there are not enough black chips in the first number. Guess what – we can rewrite the number 6 by using zero pairs. Once finished, we'll take away our 8 black chips. Whatever remains is the difference, and in this case, 6 – 8 = – 2. Here's the picture that you would draw.




**<u>EXPLORE!</u>** Try a few on your own – draw a picture for each and indicate the putting in (addition) or taking away (subtraction).

A) ** (– 7) – (– 3)

D) ** 4 + (– 7)

B) (– 3) – (– 5)

E) 7 – (– 2)

C) 4 – 7

F) 7 + 2

The last four problems are included on purpose. Did you notice the result from 4 – 7 and 4 + (– 7)? What about 7 – (– 2) and 7 + 2? Did you end up with the same result?

Let's try to come up with a general statement that will summarize what we think is happening. Make a **conjecture** or an educated and informed guess about how to rewrite $a - b$.

$a - b =$ _______________

**For Love of the Math:** *In mathematics, it isn't enough to see a pattern and then trust that it will always work. Mathematicians work to show that the pattern will ALWAYS continue by proving (or showing) that no matter what numbers are chosen, the result will follow. Here's how mathematicians would prove the conjecture made previously.*

*Claim:* $a - b = a + (-b)$

Proof:
We know that $a - b = c$, from the definition of subtraction, means that $c + b = a$. One thing we can do with equations is add something to both sides, which will maintain the equality. So we can add something like the additive inverse of $b$, which is $(-b)$.

$c + b = a$ (starting equation)

$c + b + (-b) = a + (-b)$ (adding the additive inverse to both sides)

$c + 0 = a + (-b)$ (the additive inverse property)

$c = a + (-b)$ (the additive identity property)

The last equation shows one way to represent $c$, but the definition showed $a - b = c$. Since $c$ is unique, the two quantities must be the same… which means $a - b = a + (-b)$.

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This concept is quite important as it shows another way to think of subtraction in terms of addition. Let's use this new concept to practice rewriting subtraction problems.

**EXPLORE!** Rewrite all subtractions problems using addition, but do *not* compute the result.

A) ** $419 - 27$

B) $-17 - 93$

C) $-27 - (-43) - 12 - (-17)$

D) $27 - (-43)$

# 3.13: Subtracting Integers (without chips)

Without chips, the concept remains the same – take away! And in order to remain consistent with the algorithms we've shown already, we will present the concept as if the minuend is larger than the subtrahend. In each case, we'll try to turn the problems back into addition using the techniques from the previous sections.

Examples:

A) $-9-7$ B) $-9-(-7)$ C) $9-(-7)$

Solutions:

A) For this problem, start by rewriting the subtraction as adding the opposite: $-9-7=-9+(-7)$. Now see that we have like objects, both negatives. We know that the sign will be negative, and to find the size, we add the sizes of both addends: 9 + 7 = 16. This shows why $-9-7=-9+(-7)=-16$. As it turned out, the subtraction was completed by doing addition!

B) For this problem, start by rewriting the subtraction as adding the opposite: $-9-(-7)=-9+7$. These addends are not like objects, so we'll need to find the sign on the difference. The sign will be negative since – 9 is has the bigger size; the size of the difference is found by 9 – 7 = 2. This shows why $-9-(-7)=-9+7=-2$. In this case, subtraction became addition, and we solved the addition problem by doing subtraction!

C) For this problem, start by rewriting the subtraction as adding the opposite: $9-(-7)=9+7$. Once finished, we have like objects and can quickly show that $9-(-7)=9+7=16$. As it turned out, the subtraction was completed by doing addition!

Let's summarize what these examples show. When subtracting integers, we rewrite subtraction as addition, if necessary. But what about when we are adding integers?

| | | | |
|---|---|---|---|
| A) ** | When adding integers with the same sign… | like with 13 + 47 or (– 6) + (– 8) | |
| B) | When adding integers with different signs… | like with (– 6) + 8 or 15 + (– 8) | |

What you might notice is that sometimes we do subtraction and sometimes addition, but that the values dictate what we do. In order to see this in practice, let's try a few more problems.

**EXPLORE!** For these, we want you to see how to rewrite the problem as addition or subtraction of positive numbers – focus on the sign first, then decide what subtraction or addition problem you need.

| | Original Problem | Sign (circle one) | Rewrite as a new problem |
|---|---|---|---|
| A) ** | $68-195$ | Positive Negative Zero | |
| B) | $-539-(-67)$ | Positive Negative Zero | |
| C) | $-539-67$ | Positive Negative Zero | |
| D) | $195-168$ | Positive Negative Zero | |

**EXPLORE!** Use what you learned above to calculate the actual values:

A) ** $68-195$

C) $-539-67$

B) $-539-(-67)$

D) $195-168$

**NOTE:** In part (G), some of you may just want to do subtraction directly since it is quicker. Truly, this is easier. Remember that you are in charge of these problems – if you can do the subtraction quickly, go right ahead!

But when in doubt… write it out!

Now we can put it all together. We will rewrite all subtractions to turn the entire problem into additions, then combine all like terms, and then perform the final computation.

Example: $-27-(-43)-12-(-17)$

*Step 1:* We'll rewrite the expression using addition only.

$$-27-(-43)-12-(-17)=-27+43+(-12)+17$$

*Step 2:* We'll combine the like terms – positives with positives and negatives with negatives.

$$-27-(-43)-12-(-17)=-27+43+(-12)+17=(-39)+60$$

*Step 3:* We'll perform the final computation. With some positives and negatives, the end sign will be positive (can you see why?) and the size is found by 60 – 39 = 21.

$$-27-(-43)-12-(-17)=-27+43+(-12)+17=(-39)+60=21$$

**EXPLORE!** Try some of these for yourself.

A) ** $-5-(-3)+(-7)+6$

B) $-15+(-12)-(-8)+6$

C) $9-6+11-(-3)+(-7)$

D) $-23-(-13)-(-5)-8$

When a subtraction problem is presented you, it is up to you to determine whether you want to do it as subtraction or rewrite it as addition of the opposite (additive inverse).

The additive inverse is not just with numbers, it can be used with variables too. This can help when a problem is challenging like $3x+5-(2x+1)$. In order to make this problem simpler, we need

Example: Find the additive inverse of 13*x*.

The additive inverse is the number we add to 13*x* in order to obtain 0. With like terms, we need to find the additive inverse of 13, which is – 13. So $13x+(-13x)=0$, and the additive inverse of 13*x* is – 13*x*.

Example 2: Find the additive inverse of *x* + 1.

Since the additive inverse of 1 is – 1, and the additive inverse of *x* is – *x*, then we could try adding these two pieces and see what happens; the goal is to add something to *x* + 1 and have a sum of 0.

$x+1+(-x)+(-1)=x+(-x)+1+(-1)=0+0=0$.

So the additive inverse is $(-x)+(-1)$. But the definition of the additive inverse of $x+1$ is $-(x+1)$. The fact that there are two quantities that sum to 0 with *x* + 1, means those quantities are equal.

Whenever we find an additive inverse of a quantity like this, we can use $-(x+1)=(-x)+(-1)$

**Explore:** Find the additive inverse (opposite) of the following:

A) $7x$

C) $x+2$

B) $-9x$

D) $x-9$

**Explore:** Explain in words what the expression means, and find the value.

A) ** $-(5x)$

B) $-(5+x)$

C) $-(2x-9)$

# 3.14: Number Sense and Addition/Subtraction

When we add numbers, many people think that the size goes up. And in one manner of thinking, it does: $27+9$ will be larger than 27.

The size of the sum or difference depends on the numbers involved, and the key idea to understand relates to 0 again. $(-8)+0=-8$, so adding 0 doesn't change the value because of the additive identity property. What about $(-8)+(-6)$? When we add those numbers, we find $(-8)+(-6)=-14$ which is actually *smaller* than – 8. Adding a number greater than 0 will make the result bigger, and adding a number less than 0 will make the result smaller.

With addition or subtraction, we start with one value and then either add to it, or take something away. But the value added or subtracted affects the sum or difference. Earlier, we discussed how the sign and size affect the result of addition and subtraction. Our rules are summarized again here:

- Subtraction – We can rewrite subtraction as adding the opposite and follow the rules for addition.
- Addition – We have two cases: (1) same sign addends, and (2) different sign addends.

    1. Same sign addends: we add the sizes and keep the sign.

    2. Different sign addends: we subtract the sizes (always as big – small) and keep the sign of the addend with larger size.

These rules give us a way to compute values, but it doesn't give much insight into adding or subtracting from a number. When we start with a number, and add a positive number, will the result be more or less than what we started with? What if we added a negative number?

Interactive Examples: circle whether the sum/difference is greater or less than the starting value.

A) $8+6$ will be… greater / less than 8.

B) $8+(-12)$ will be… greater / less than 8.

C) $14-5$ will be… greater / less than 14.

D) $14-(-7)$ will be… greater / less than 14.

To summarize the results, we can make a list of what we saw with addition:

| | **If B is…** | **Then A + B will be …** | | |
|---|---|---|---|---|
| A) | 0 | Greater than A | Less than A | Equal to A |
| B) ** | Positive | Greater than A | Less than A | Equal to A |
| C) | Negative | Greater than A | Less than A | Equal to A |

To summarize the results, we can make a list of what we saw with subtraction:

| | If B is… | Then A – B will be … |
|---|---|---|
| D) | 0 | Greater than A Less than A Equal to A |
| E) ** | Positive | Greater than A Less than A Equal to A |
| F) | Negative | Greater than A Less than A Equal to A |

**EXPLORE!** Determine the size and sign of the quotient without performing any operations.

| | Operation | Resulting Sign | Value is… |
|---|---|---|---|
| G) | $(-12)-(-18)$ | Pos Neg Zero | Greater than – 12 Less than – 12 |
| H) ** | $(-12)+(-5)$ | Pos Neg Zero | Greater than – 12 Less than – 12 |
| I) | $(-12)+(6.5)$ | Pos Neg Zero | Greater than – 12 Less than – 12 |
| J) | $(-12)-(6.5)$ | Pos Neg Zero | Greater than – 12 Less than – 12 |
| K) | $(-12)-(-12)$ | Pos Neg Zero | Greater than – 12 Less than – 12 |
| L) | $(23)-(-14)$ | Pos Neg Zero | Greater than 23 Less than 23 |
| M) | $(23)+(14)$ | Pos Neg Zero | Greater than 23 Less than 23 |
| N) | $(23)-(149)$ | Pos Neg Zero | Greater than 23 Less than 23 |
| O) | $(23)+(-49.7)$ | Pos Neg Zero | Greater than 23 Less than 23 |

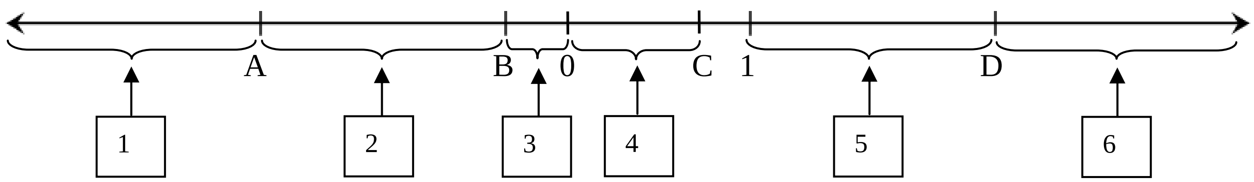
The goal of these types of questions is to help develop “number sense.” We’ll put your number sense to the test with number lines next!

# 3.15: Subtracting with Number Lines

Since subtraction can be rewritten as addition, doing subtraction on the number line is nearly identical to doing addition with the number line.

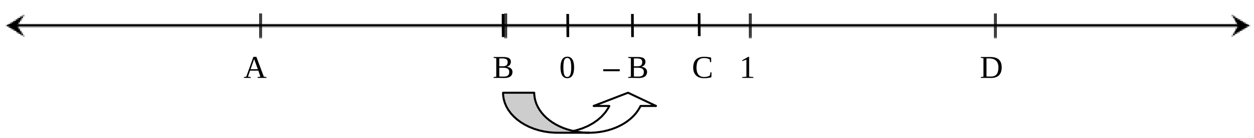
Here is our standard number line for these problems.


Then we put in the regions for our answers:

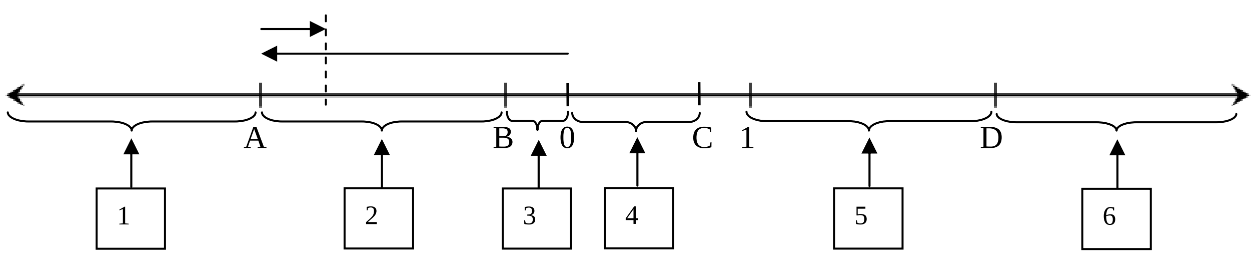



<u>Example:</u> Find the region that best approximates A – B.

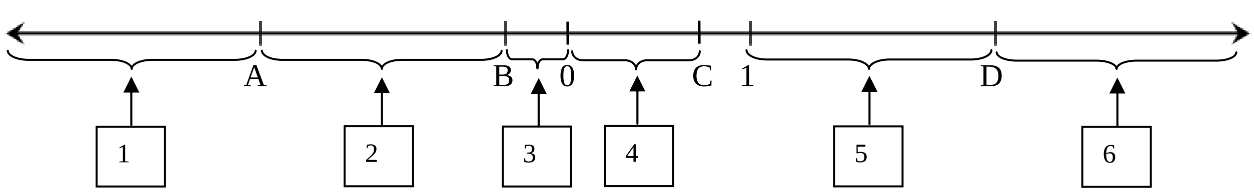
*Step 1:* To find the value, we can rewrite A – B as A + (– B), and if we find (– B), it is fairly quick.




*Step 2:* Using our number sense, we know that if B is negative, then A – B is bigger than A. Move the size of B above A. This puts A – B into region 2!




<u>Interactive Example:</u> Find the region that best approximates B – (– 1).



**EXPLORE!** Use a number line to complete the subtraction.

A) ** Find the region that best approximates B – A.

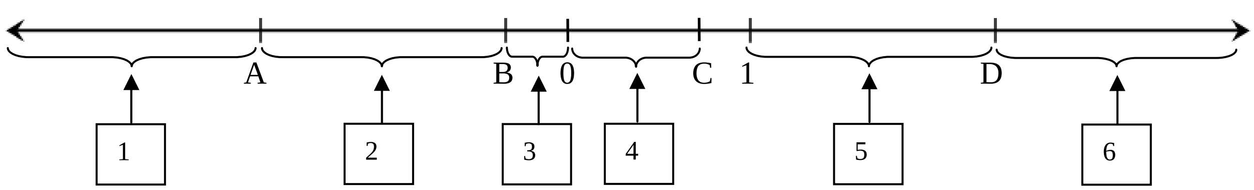
B) Find the region that best approximates 2 – D.

C) Find the region that best approximates C – 1.

D) Find the region that best approximates B – C.

E) Find the region that best approximates C – B.

F) Find the region that best approximates 1 – D.



# 3.16: Subtracting Fractions, Decimals, and More

The concept of rewriting subtraction as addition makes subtraction with all other numbers and objects exactly the same.

- For fractions, when denominators are different, we use FLOF to make them the same. When denominators are the same, we just add or subtract the numerators.
- For decimals, we line up decimal places so that we're adding or subtracting like objects.
- For all other objects, we combine like objects and adjust the sign based on the coefficient.

The best way to deal with these ideas is to practice.

*(Whole Numbers)*

A) $411 - 216$

C) $914 - 698$

B) $8000 - 723$

D) $9385 - 467$

*(Integers)*

A) $-38 - (-17)$

C) $-16 - |-35|$

B) $23 - 426$

D) $-9 - 7 + (-11) + 13 - (-24)$

*(Fractions)*

A) $-\frac{7}{8} + \frac{1}{4}$

C) $\frac{4}{7} - \frac{1}{3}$

B) $-\frac{2}{3} - \left(-\frac{5}{6}\right)$

D) $-\frac{8}{11} + \left(-\frac{3}{4}\right)$

*(Decimals)*

A) $30.27 - 15.8$

C) $-4.2 - (-5.9)$

B) $-1.035 - 0.987$

D) $2.3 - 42.6$

*(Mixed Bag)*

A) 7 tables – 3 desks =

C) $-23x + (-7y) - 9y - (-18x)$

B) $13x - (-7y) - 9y + (-8x)$

D) $-3\sqrt{7} + 5\sqrt{2} - (-8\sqrt{7}) - 19\sqrt{2}$

*(Word problem)* Louis has $1,862 in his bank account. He purchased a fancy tablet computer for $979. How much money would Louis have left in his account?

# 3.17: Estimation, Property Review, and Magic Boxes

When we work with addition and subtraction using a calculator, it's very good to have the number sense to **estimate** (or approximate) the value. Estimating is a way to get close to the answer quickly. For estimating, we don't care about what the actual value is, but just about getting close to it.

| | Estimating the value of … | Is closest to… |
|---|---|---|
| A) ** | 64.9184 + 3.209 | 60 68 97 9.7 |
| B) ** | $\frac{2}{3}+\frac{1}{5}$ | $\frac{3}{8}$ $\frac{1}{5}$ 0 1 |
| C) | 115 – 123 | 10 0 – 10 – 220 |
| D) | – 115 – ( – 123) | 10 0 – 10 – 220 |
| E) | – 115 – ( – 108) | 10 0 – 10 – 220 |
| F) | – 62 – 33 | 30 – 30 – 90 90 |
| G) | 135.4738 + 2.54 + 0.0694 | 140 14 1.4 1,400 |
| H) | 147.204 – 39.0294 | 110 11 1.1 1,100 |
| I) | $\frac{2}{5}-\frac{1}{3}$ | $\frac{3}{8}$ $\frac{1}{2}$ 0 1 |

We could also look at some use of the properties of addition and subtraction to see if we think these are correct or incorrect uses of a property.

| | Property Name | Using it like this… | Is… |
|---|---|---|---|
| J) ** | Additive Inverse | $5+3+0=5+3$ | Incorrect Correct |
| K) ** | Additive Identity | $0+473=473$ | Incorrect Correct |
| L) | Commutative Property | $45-19=19-45$ | Incorrect Correct |
| M) | Associative Property | $5-(3+7)=(5-3)+7$ | Incorrect Correct |
| N) | Commutative Property | $5+(3+7)=(5+3)+7$ | Incorrect Correct |
| O) | Additive Inverse | $3+0=3$ | Incorrect Correct |

As Monty Python would say… "And now for something completely different!"

We'll use these directional boxes quite a bit coming up, so it's good to get a start with them now. The plan is to fill in the missing boxes and follow the directions.

<u>Example:</u>

[ 1 ] —(+ 3)→ [   ] —(– 4)→ [   ]

Start with the 1 and then add 3, write the result in the middle box. Then take that number, 4, and subtract 4 to get the number for the final box. When finished, it should look like this:

[ 1 ] —(+ 3)→ [ 4 ] —(– 4)→ [ 0 ]

**<u>EXPLORE!</u>** Try some more – fill in all the other boxes. You may have to work backwards!

A) [ – 5 ] —(– 7)→ [   ] —(– 4)→ [   ]

B) [ – 5 ] —(+ 9)→ [   ] —(– 13)→ [   ]

C) [ 12 ] —(– 43)→ [   ] —(– (–5))→ [   ]

D) [ 14 ] —(+ 17)→ [   ] —(– 67)→ [   ]

E) [   ] —(– 7)→ [ – 5 ] —(– 4)→ [   ]

F) [   ] —(+ 9)→ [ – 5 ] —(– 13)→ [   ]

G) [   ] —(– 43)→ [   ] —(– (–5))→ [ 12 ]

I) [   ] —(+ 17)→ [   ] —(– 67)→ [ 14 ]

Summarize what you've seen:

- If you started with "+ 17" over an arrow and had to go backwards, what would you do?

- If you started with "– 8" over an arrow and had to go backwards, what would you do?

# 3.18: Subtraction Summary

If you recall from addition, there were a number of properties that held true. But, do those same properties work with subtraction?

- Is subtraction commutative? Write a few equations that would be true if subtraction was commutative, then explain your conclusion.

- Is subtraction associative? Write a few equations that would be true if subtraction was associative, then explain your conclusion.

To summarize the results, we can make a list of what we saw with subtraction:

| | **If B is…** | **Then A – B will be …** | | |
|---|---|---|---|---|
| A) | 0 | Greater than A | Less than A | Equal to A |
| B) ** | Positive | Greater than A | Less than A | Equal to A |
| C) | Negative | Greater than A | Less than A | Equal to A |

**Concept Questions**

When dealing with integers, the sign of the sum depends on the value of the addends. The sum could be always positive (P), always negative (N), or sometimes positive and sometimes negative (S). Label each of the following expressions as P, S, or N. If the answer is P or N, explain why. But if the answer is S, give one example that shows a positive result and one example that shows a negative result.

| | **Expression** | **Sign (circle one)** | **Examples or Explanation** |
|---|---|---|---|
| D) | pos – pos | P S N | |
| E) | neg – pos | P S N | |
| F) | pos – neg | P S N | |
| G) | neg – neg | P S N | |
| H) | pos + pos | P S N | |
| I) | pos + neg | P S N | |
| J) | neg + pos | P S N | |
| K) | neg + neg | P S N | |