Math Fundamentals for Statistics (Math 52)

Unit 4: Multiplication

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"The 'How' and 'Whys' Guys"

4.1: Multiplication of Whole Numbers

Multiplication is another main operation used in mathematics at nearly all levels. At different points in your life, someone may have used the word "times" to reference multiplication. Let's see where that comes from.

With addition, think about how you would compute: 3 + 3 + 3 + 3 + 3. We would end up with 15, since we would add 3 five times. Yes, we add five times, and we added the number 3, which is a way to represent five times three. In a sense, it's a shortened form of saying five times (we added) three. It is often written as 5×3 or $5 \cdot 3$ or $5(3)$.

One way to attempt to define multiplication of whole numbers is as this type of repeated addition.

$$\underbrace{a + a + a + \ldots + a}_{n \text{ times}} = n \times a\,.$$

Our definition is excellent, except for one issue: $0 \times a$. The definition for writing a sum of $0 \times a = \underbrace{a + a + a + \ldots + a}_{0 \text{ times}}$ simply doesn't work – since we can't define what "0 times" means. Since another product, $n \times 0$, is well defined as $n \times 0 = \underbrace{0 + 0 + 0 + \ldots + 0}_{n \text{ times}} = 0$, it would make sense to define the result of $0 \times a$ to be the same value. The following complete definition brings all of this together for whole numbers:

Definition: The multiplication of whole numbers *a* and *b* is written $a \times b$. In a multiplication equation like $a \times b = c$, *a* and *b* are called **factors** and *c* is called the **product**.

$$n \times a = \begin{cases} \underbrace{a + a + a + \ldots + a}_{n \text{ times}} & \text{if } n > 0 \\ 0 \quad \text{if } n = 0 \end{cases}$$

For Love of the Math: *In higher level mathematics where the number sets are examined in higher detail, there are words describing the real numbers as a* ***field*** *– which is a set of numbers and two operations that satisfy certain properties. For the real numbers, those two operations are addition and multiplication. Over time, it became simpler to create new operations that could be tied back to the field properties. As we will see, multiplication is a way to describe a type of repeated addition and it truly is a new operation!*

Any new operations that we have in mathematics should be explored – how can we write it correctly? Are there ways we could re-write it correctly? When addition was explored, we had some interesting properties: Commutative Property, Associative Property, Additive Inverse, and Additive Identity. Do any of these properties hold true with multiplication?

Let's explore these questions together.

EXPLORE! Based on our multiplication knowledge, find the following values (Split Room):

A) 7×9 (LS)

B) 9×7 (RS)

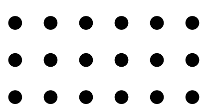
C) 8×5 (LS)

D) 5×8 (RS)

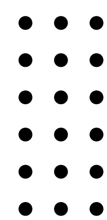
What do you notice about multiplication and the order of the factors? How could you rewrite something like $a \times b$?

This general rule is called the **Commutative Property of Multiplication:** $a \times b = b \times a$ for all numbers. You might think of this like it was a commute to work and then home – changing the order of the factors will not change the product.

The commutative property of multiplication could be represented visually. With multiplication, we can observe that $3 \cdot 6$ would represent 3 groups of 6. Visually, this could be 3 rows by 6 columns and would look like this:



Now compare this with $6 \cdot 3$? It is represented as 6 groups of 3, and visually would be 6 rows by 3 columns…



Both have the same value and tilting your head, the two images look exactly the same!

EXPLORE! Let's try another one based on our multiplication knowledge, find the following values:

A) $2 \times (3 \times 4)$ (LS)

B) $(2 \times 3) \times 4$ (RS)

C) $(2 \times 9) \times 1$ (LS)

D) $2 \times (9 \times 1)$ (RS)

What do you notice about grouping of the factors? How could you rewrite an expression like $(a \times b) \times c$?

This general rule is called the **Associative Property of Multiplication:** $(a \times b) \times c = a \times (b \times c)$ for all numbers. You might think of this like it was about the grouping – changing the grouping of the factors will not change the product.

Interactive Examples: There are a few other properties that might seem obvious, but are often helpful:

A) 1×9 B) 23×1 C) 231×1

This general rule is called the **Identity Property of Multiplication:** $a \times 1 = 1 \times a = a$ for all numbers. Anytime we multiply a number by 1, or multiply 1 by a number, the result is just the original number.

For addition, the number zero (0) had a very special place in mathematics and was called the **Additive Identity Element**. With multiplication, the number one (1) has that very special place in mathematics and is called the **Multiplicative Identity Element**.

For Love of the Math: *The identity elements are critical to the operations using them. For addition, whether a number is positive or negative (compared to 0) determines whether the result is bigger or smaller than the original number. Also, adding the identity element itself would not change the value. Likewise, as we'll see in this unit, the size of a number being larger than 1 or smaller than 1 determines the same type of result with multiplication. If the size is larger than 1, the result will be larger than the original number. If the size is smaller than 1, the result will be smaller than the original number, and if the size is equal to 1, then the product won't change.*

Addition can make numbers larger or smaller depending on the number being positive or negative. Multiplication can make numbers larger or smaller depending on size: greater than 1 or less than 1.

Finally, let's determine how multiplication works with other operations like addition. For this one, an example can be helpful.

A shipping company has a way of packing their items together – each large crate will contain 2 tables and 7 chairs. When a customer orders 3 crates, how many items of each type are ordered?

What is interesting here is that we could represent the quantity in two different ways:

- $3(2 \text{ tables} + 7 \text{ chairs})$
- 6 tables + 21 chairs

Since there were 3 crates, there would be 3 groups of 2 tables each, and 3 groups of 7 chairs each. When items are grouped with addition and then multiplied, the result can be represented as:

$$3(2 \text{ tables} + 7 \text{ chairs}) =$$
$$(2 \text{ tables} + 7 \text{ chairs}) + (2 \text{ tables} + 7 \text{ chairs}) + (2 \text{ tables} + 7 \text{ chairs}) =$$
$$3 \cdot (2 \text{ tables}) + 3 \cdot (7 \text{ chairs})$$

This is a way of rewriting and regrouping all at the same time. Now let's try another one based on our multiplication knowledge.

<u>Example:</u> Rewrite $3(x+8)$ without any parenthesis:

$$3(x+8)=(x+8)+(x+8)+(x+8)=3x+3\cdot 8=3x+24$$

<u>Interactive Exercise:</u> Try one more - rewrite $7(2x+3m)$ without any parenthesis:

$7(2x+3m)$

This general rule is called the **<u>Distributive Property of Multiplication over Addition:</u>** $a(b+c)=ab+ac$ for all numbers. This property is really a way of demonstrating a short-cut in the process of rewriting the product – instead of writing the addition repeatedly and then combining the like terms, we could distribute the multiplication to each of the terms (addends). The distributive property is what happens when multiplication and addition come together.

<u>Interactive Example:</u> Why do we <u>*not*</u> need to define this property over subtraction?

<u>EXPLORE!</u> Now try a few more to practice this new idea – rewrite without any parenthesis:

A) $4(3x+6m)$

B) $9(y-4+2n)$

C) $5(x+2y-7)$

D) $4(30+5)$

For some problems like $4(30+5)$, there are two ways to find the result: (1) you could add 30 + 5 and then multiply by 4, or (2) distribute the 4 and then add the two resulting products. But for $6(x+5)$, since the *x* and 5 are not like terms, there is only one way to rewrite this without parenthesis.

With arithmetic sequences $a_n = a + (n-1)d$, we could use this technique to simplify the expressions.

EXPLORE! Find the n^{th} term in the arithmetic sequence, and simplify the result.

A) ** Arithmetic sequence with $a = 15$ and $d = 3$.

B) Arithmetic sequence with $a = 9$ and $d = 2$.

C) Arithmetic sequence with $a = 23$ and $d = 5$.

D) Arithmetic sequence with $a = -3$ and $d = 4$.

EXPLORE! Let's practice the properties.

- Commutative Property of Multiplication: $a \times b = b \times a$
- Associative Property of Multiplication: $(a \times b) \times c = a \times (b \times c)$
- Identity Property of Multiplication: $a \times 1 = 1 \times a = a$
- Distributive Property of Multiplication: $a(b + c) = ab + ac$

A) ** $7 \times 3 = 3 \times 7$

B) $7 \times (3 \times 5) = (7 \times 3) \times 5$

C) $7 \times (3 \times 5) = 7 \times (5 \times 3)$

D) $7 \times 1 = 7$

E) $7(8 + 2) = 7 \times 8 + 7 \times 2$

F) $25 \times (37 \times 4) = 25 \times (4 \times 37) = (25 \times 4) \times 37$

As we begin to move into computing larger products and doing some multiplication, we'll need a tool that comes from place value. Remember that our place values are in groups of ten (10). Ten ones make one group of ten… ten groups of ten makes one group of hundred… ten groups of hundred makes one group of thousand. When we write these place value facts out in multiplication notation, a nice pattern emerges:

- $10 \times 1 = 10$
- $10 \times 10 = 100$
- $10 \times 100 = 1{,}000$
- $10 \times 1{,}000 = 10{,}000$

What pattern do you see from these examples?

Multiplying by 10 can be very quick, and similarly with multiplying by 100, or more.

<u>EXPLORE!</u> Practice a few below – multiply using place value numbers:

A) $^{**}1{,}000 \times 100 =$

C) $167 \times 100 =$

B) $10{,}000 \times 1{,}000 =$

D) $100 \times 8{,}000 =$

We could even use the associative property to multiply quickly with groups of 10.
An example would be $8 \times 60 = 8 \times (6 \times 10) = (8 \times 6) \times 10 = 48 \times 10 = 480$.

<u>EXPLORE!</u> Try a few of this new type:

A) ** $120 \times 300 =$

C) $70{,}000 \times 600 =$

B) $80 \times 300 =$

D) $13 \times 2{,}000 =$

The distributive property can also be used to simplify calculations… going one direction or another.

Examples: A) 7×14 and B) 6×23

A) Using the distributive property requires addition, so let's rewrite 14 using addition, and then distribute: $7\times 14 = 7(10+4) = 7(10)+7(4) = 70+28 = 98$
B) Using the distributive property requires addition, so let's rewrite 23 using addition, and then distribute: $6\times 23 = 6(20+3) = 6(20)+6(3) = 120+18 = 138$

EXPLORE! Try a few on your own to see that multiplying can be done mentally using the distributive property with some numbers very quickly. First rewrite, then use the distributive property as done above.

A) ** 87×3

B) 38×4

C) 6×27

Other examples allow us to use the distributive property to add like groups (objects). This can save time – instead of doing separate multiplication problems and then adding the results, we could turn the problem into just one multiplication. We'll work backwards – *so turn your chairs around*!

Examples:

A) 8 dozen added to 6 dozen is:

B) 73 groups of 46 + 5 groups of 46 is:

Further Examples: $83\times 17+83\times 13$

$83\times 17+83\times 13$ really means 83 groups of 17 added to 83 groups of 13. Since both sections have 83 groups, we could combine: $83\times 17+83\times 13 = 83\times(17+13) = 83\times 30$.

EXPLORE! Try rewriting a few on your own to see that multiplying and adding can sometimes be combined to save time. For these problems, you only need to re-write as one multiplication (instead of two or more).

A) ** $3\times 80+3\times 7$

B) $38\times 4+38\times 8$

C) $54\times 19+21\times 19$

D) ** $-46\times 17+17\times 10$

E) $38\times 46-24\times 46$

F) $231\times 29+9\times 29+29\times 107$

As a way to make things easier, different multiplication algorithms were invented – all based on the ideas and concepts we've described to this point. For the following types of algorithms, we'll start each one with the same problem, 37×85. This way you can see similarities and differences between the types of algorithms.

Algorithm #1: **USING PROPERTIES AND MULTIPLYING BY 10**

$$\begin{aligned}
37 \times 85 &= \\
(30+7)\times(85) &= && \text{Rewriting 37 using addition... } 37 = 30 + 7 \\
(30)\times(85) + 7\times(85) &= && \text{Applying the Distributive Property} \\
(30)\times(80+5) + 7\times(80+5) &= && \text{Rewriting 85 using addition... } 85 = 80 + 5 \\
(30)\times(80) + (30)\times(5) + 7\times(80) + 7\times(5) &= && \text{Applying the Distributive Property} \\
2400 + 150 + 560 + 35 &= && \text{Multiplying with place values (10s, 100s, etc)} \\
3145 & && \text{Adding the resulting numbers to find the result}
\end{aligned}$$

While mathematicians might enjoy this process a lot, for most people using the properties in this way becomes very tedious and quicker (or more efficient) ways were desired. However, this algorithm does show how to get the result without having to write out 85 + 85 + 85 + 85… + 85 a total of thirty-seven times. This is what it looks like to not skip any steps at all. *Please, avoid this for your sanity!*

Algorithm #2: **PARTIAL PRODUCTS**

In this algorithm, we write the results vertically instead of horizontally, and examine each product in order (remembering the place values).

$$\begin{array}{rl}
85 & \leftarrow \text{Original Factor} \\
\underline{\times 37} & \leftarrow \text{Original Factor} \\
35 & \leftarrow 7\times 5 \\
560 & \leftarrow 7\times 80 \\
150 & \leftarrow 30\times 5 \\
\underline{2400} & \leftarrow 30\times 80 \\
3{,}145 & \leftarrow \text{Final Product}
\end{array}$$

What we can see from this is a slightly quicker way to do the multiplication, but it still uses the distributive property and place value multiplication. In fact, this algorithm is the first main step between the pure mathematical way and the traditional algorithm.

Algorithm #3: **TRADITIONAL ALGORITHM**

In this algorithm, we write the results vertically again, but instead of writing down all of the partial products, we add some of them to make the result less "bulky." For the big picture, this algorithm uses the distributive property as we did in some examples.

The way we use the distributive property to find the product is to find separate groups and then add and combine groups: $7\times 85 + 30\times 85 = (7+30)\times 85 = 37\times 85$.

The process in order is on the following page:

Step 1: Start by writing the problem vertically and draw a line under the bottom.

Step 2: Start by multiplying the ones: $7 \times 5 = 35$. Think of this as 3 tens and 5 ones, and write the 3 groups of ten above the tens in the top factor. Then write the 5 ones in the ones place under the line.

Step 3: Now multiply the 7 ones by 8 tens, which is $7 \times 80 = 560$. Since 560 is 56 tens, we combine these 56 tens with the previous 3 tens (regouped), and we write that final value (59) in the tens place under the line. *If there were more digits in the factor, we would regroup hundreds and repeat the process.* By combining these two together, we have just done $7 \times 5 + 7 \times 80$, which is 7×85.

Step 4: Now consider the next digit in the 37… the 3, or 3 groups of ten. We'll begin by multiplying the 3 tens by 5: $30 \times 5 = 150$. Think of this as 1 hundred and 5 tens. Write the 1 hundred above the next digit in the top factor and write the 5 tens in the tens place under the line as 50.

Step 5: Now multiply the next digits, the 3 tens and 8 tens. $30 \times 80 = 2400$, but think of this as 24 hundreds. Combine 24 hundreds with the previous 1 hundred (regrouped) to get 25 hundreds. Write this below the line in the hundreds place. *If there were more digits in the factor, we would regroup hundreds and repeat the process.* By combining these two together, we have just done $30 \times 5 + 30 \times 80$, which is 30×85.

Step 6: The last step in the algorithm is to add together the previous products. Write the result under the last line and we have shown that $37 \times 85 = 3145$. The addition is needed because $7 \times 85 + 30 \times 85 = 37 \times 85$.

Step 1	***Step 2***	***Step 3***	***Step 4***	***Step 5***	***Step 6***
85 $\underline{\times 37}$	3 85 $\underline{\times 37}$ 5	3 85 $\underline{\times 37}$ 595	1 3 85 $\underline{\times 37}$ 595 50	1 3 85 $\underline{\times 37}$ 595 $\underline{2550}$	1 3 85 $\underline{\times 37}$ 595 $\underline{2550}$ 3145

As seen, this links nicely to the partial products algorithm, but is written with less vertical space.

$$\begin{array}{rl} 85 & \leftarrow \text{Factor} \\ \underline{\times 37} & \leftarrow \text{Factor} \\ 595 & \leftarrow 7 \times 85 \\ \underline{2550} & \leftarrow 30 \times 85 \\ 3{,}145 & \leftarrow \text{Product}: \ 7 \times 85 + 30 \times 85 = 37 \times 85 \end{array}$$

The traditional algorithm is probably one of the most efficient, but it does combine addition into the first steps instead of just doing multiplication. As a result, sometimes students who are learning this can become lost in the process. Remember the partial products algorithm as a way to step back.
EXPLORE! Which algorithm would you use to compute these products?

	Product	Partial Products	Traditional Algorithm
A) **	18×36		
B)	47×39		
C)	79×85		
D)	435×72		

EXPLORE! Now compute the actual products.

A) 18×36

C) 79×85

B) 47×39

D) 435×72

Estimating is a way to get close to the answer quickly. For estimating, we don't care about what the actual value is, but just about getting close to it.

	Estimating the value of …	Is closest to…
A) **	$3{,}547 \times 219$	70,000 7,000 700,000 7,000,000
B)	$1{,}957 \times 3{,}219$	6,000,000 600,000 60,000 6,000
C)	7×499	350 3,500 35,000 350,000

Now use your calculator to find the actual value

A) 🖩	$3{,}547 \times 219$	
B) 🖩	$1{,}957 \times 3{,}219$	
C) 🖩	7×499	

Estimating helps us to spot if we typed something into the calculator incorrectly, which might save us points on an exam!

Algorithm #4: **DISTRIBUTIVE PROPERTY FOR SPEED**

In this algorithm, we rewrite a factor in order to multiply faster – and the results often allow us to multiply mentally instead of with paper/pencil or a calculator.

Many numbers are already easy to multiply: $3 \times 4{,}000$. So if we ended up with factors that are close to the easy ones, we'll just rewrite them.

Example: $3 \times 3{,}999$ seems like a more challenging problem until we rewrite the 3,999 as 4,000 – 1.

$$3 \times 3{,}999 = 3(4{,}000 - 1) = 12{,}000 - 3 = 11{,}997$$

Example #2: For larger numbers, just do the subtraction slowly. 17×498 seems like a more challenging problem until we rewrite the 498 as 500 – 2.

$17 \times 498 = 17(500 - 2) = 17(500) - 17(2) = 8{,}500 - 34$. To do this quickly, start by subtracting 30, then subtract 4. In your mind, you can count back: 8,500 – 30 = 8,470, then 8,470 – 4 = 8,466.

EXPLORE! Try a few more on your own using the distributive property.

A) ** 8×697

C) 14×801

B) ** 11×507

D) 21×399

4.2: Multiplication of Integers

Multiplication of integers is extremely similar to whole numbers, but there are a few things that look different. One of the most important ideas is the additive inverse (or opposite). We could write -3 as the additive inverse of 3: $-3 = -(3)$.

Interactive Examples:

A) Considering multiplication as the repeated addition, try to find one way we could rewrite the following multiplication using addition: $5 \times (-3)$.

B) What about $4 \times (-7)$? Rewrite this as addition, then find the result.

C) One more… try $(-2) \times 4$? Rewrite this as addition, then find the result.

For this last one, we could consider it as $(-2) \times 4$ or as $-(2) \times 4$… the opposite of 2×4. Since $2 \times 4 = 8$, $-(2) \times 4 = -8$. And since multiplication is commutative, we could also write this as $(-2) \times 4 = 4 \times (-2) = (-2) + (-2) + (-2) + (-2) = -8$. Multiple ways to get the same answer (pardon the pun)!

Interactive Examples: But what happens when both factors are negative… like with $(-8)(-2)$? It may help to review some previous concepts about addition.

A) What is the result of $2 + (-2)$?

B) How about $7 + (-7)$?

C) What is the result of $6(0)$?

D) How about $(-8)(0)$?

These are essentially zero pairs – two numbers that are additive inverses will always sum to 0. And as with the definition of multiplication, any number multiplied by 0 is 0. You may be wondering how this relates to multiplication, and the key is the distributive property!

We'd like to show you how mathematicians came up with the properties and 'rules' about multiplying two negatives, and we'll do it with a reasonably quick proof. Take a look and as you read through it, try to follow the reasoning that moves from one step to another. Can you follow the reasoning?

$(-8)(-2)=$	
$-(8)(-2)=$	*Use the additive inverse to rewrite* $(-8)=-(8)$
$-(-16)=$	*Multiply the positive by the negative:* $(8)(-2)=-16$
16	*Use the additive inverse to rewrite* $-(-16)=16$

So what we have is really that the product of two negatives is the opposite of a negative, which we already know is positive! This shows "why" the product of two negatives is positive instead of just having to memorize it.

EXPLORE! Prove that $(-7)(-5)=35$ using steps similar to the ones above.

EXPLORE! Try a few with products where we don't have to use the proof – we can just use the result. And be sure that your product has the appropriate "sign":

A) $(-8)(7)$

C) $(-8)(-11)$

B) $(-9)(13)$

D) $(5)(-9)$

Concept Questions

When dealing with integers, the sign of the result depends on the sign of the starting numbers. The result could be always positive (P), always negative (N), or sometimes positive and sometimes negative (S). Label each of the following expressions as P, S, or N. If the answer is P or N, explain why. But if the answer is S, give one example that shows the product could be positive and one example that shows the product could be negative. *Note: some questions review previous topics as well as multiplication.*

	Expression	**Sign (circle one)**	**Examples or Explanation**
A) **	pos × pos	P S N	
B)	pos × neg	P S N	
C)	neg × neg	P S N	
D)	neg × pos	P S N	
E)	pos + neg	P S N	
F)	neg + neg	P S N	
G)	neg – pos	P S N	
H)	neg – neg	P S N	
I)	pos – neg	P S N	
J)	pos – pos	P S N	

Which of the operations seem the easiest to understand and explain (addition, subtraction, or multiplication)? Why do you think it is easier?

Important ideas: *Size and sign*.
When we multiply integers, the size is found by multiplying the sizes of the factors.

When we multiply integers with the same sign, the product is… (positive or negative).

When we multiply integers with different signs, the product is… (positive or negative).

In Unit 2, we saw the additive inverse being used to rewrite expression. $-(x-5)=-x+5$. With the multiplication of integers, we can now multiply quantities by – 1. Let's see what happens when we multiply an expression by – 1.

We know that the additive inverse of 5 is – 5, so $-(5)=-5$. But from this section, we know that $(-1)(5)=-5$. Since the result from both is the same, then $-(5)=(-1)5$. If we have a quantity with additive inverses (opposites), we could rewrite it using multiplication by – 1. So $-(x)=-x$ for any number.

<u>Example:</u> Find the value of $(-1)(x-5)$ using the distributive property.

$$(-1)(x-5)=(-1)(x+(-5))=(-1)x+(-1)(-5)=-x+5$$

From Unit 2, this result matches what was previously seen: $-(x-5)=-x+5$. It shows $(-1)(x-5)=-(x-5)$. In the future, either of these techniques can be used and they are interchangeable.

<u>EXPLORE!</u> Write the following values without parenthesis.

A) ** $-(x+3)$

B) $(-1)(x+9)$

C) $-(4x-7y+(-1))$

D) $(-1)(7-x+y)$

With arithmetic sequences $a_n=a+(n-1)d$, we could use this technique to simplify the expressions.

<u>EXPLORE!</u> Find the n^{th} term in the arithmetic sequence, and simplify the result.

A) ** Arithmetic sequence with a = 5 and d = – 3.

B) Arithmetic sequence with a = – 9 and d = – 2.

4.3: Rewriting Multiplication (Factoring)

When we encountered addition, we looked to rewrite the sum using addition. Here's a quick review:

Write 30 using addition (give 3 examples): 30 = 28 + 2; 30 = 25 + 5; 30 = 17 + 13

We could also rewrite 30 using either subtraction or addition with integers: 30 = 32 – 2 = 30 + (– 2); 30 = 47 – 17 = 47 + (– 17)

This concept of rewriting a number using addition allowed us to quickly form sums or differences. Some examples of how quick this can be are:

- $700-437=(1+699)+(-437)=1+(699+(-437))=1+(699-437)=1+262=263$
- $698+437=((-2)+700)+437=(-2)+(700+437)=(-2)+(1137)=1135$

Now that we understand the basics of multiplication, we'll need to be able to write products as factors. Just like addition and subtraction, we could rewrite a number using multiplication in a process known as **factoring**. This may seem to be more challenging, but can be fairly quick and even fun! ☺

"Rewrite 30 using multiplication" will be worded as "Factor 30." In the space below, write as many different factorizations of 30 as you can.

Example: Factor 30:

- $30=1\times 30$
- $30=2\times 15$
- $30=3\times 10$
- $30=5\times 6$
- $30=2\times 3\times 5$

Interactive Example #2: Now factor 24:

	Rewrite	Addition	Subtraction	Multiplication
A) **	50			
B)	72			
C)	18			
D)	49			

Some factorizations are pretty interesting to mathematicians, because they are factorizations that cannot be further factored. For example, the only way to factor the number 2 is 1×2. When a number has exactly two different factors, 1 and itself, then that number is called **prime**. Any number that is the product of prime numbers is called **composite**. For the factorizations we're considering the natural numbers only: 1, 2, 3, 4, … (no negatives, fractions, decimals, etc.).

For Love of the Math: *In higher level mathematics where the number sets are examined in higher detail, factorization into prime numbers is extremely important. One of the great theorems (proofs) in mathematics is the* ***Fundamental Theorem of Arithmetic****, and it says, that every integer greater than 1 is either prime or a product of primes. More importantly, it says that prime factorization must be unique. So if you and I both factor a number into primes, we'll end up at the same place but maybe have the multiplication in a different order.*

But what about the number 1… is it prime or composite? Why is it not included in the Fundamental Theorem of Arithmetic?

Sure, it factors into 1×1 so it feels like it should be prime because the only factors are 1 and itself. However, $1=1\times 1$ and $1=1\times 1\times 1$ and $1=1\times 1\times 1\times 1$. That means that when factoring 1, two people would not necessarily end at the same result which means that the factorization is not **unique**. Since 1 fails the unique factorization portion of the theorem, 1 cannot be prime. But since 1 is not the product of primes, then 1 cannot be composite. That's right, 1 is the only positive integer that is neither prime nor composite. How quaint!

There are song lyrics that say "One is the loneliest number that you'll ever do." Why would you say that "one" the loneliest number in factoring?

The song actually continues to say: "Two can be as bad as one, it's the loneliest number since the number one." This is eerily true, because 1 is neither prime nor composite, but 2 is prime. And 2 is even. No other even number can be prime, so the number two is really all alone as the only even prime number. Which is fairly odd! ☺

EXPLORE! Describe the following numbers as prime (P), composite (C), or neither (N) and give one factor other than 1:

	Number	Describe	A factor…
A) **	28	P C N	
C)	23	P C N	
E)	16	P C N	
G)	1	P C N	
I)	17	P C N	
K)	81	P C N	

	Number	Describe	A factor…
B) **	44	P C N	
D)	9,003	P C N	
F)	11	P C N	
H)	35	P C N	
J)	605,930	P C N	
L)	0	P C N	

How do we know if larger numbers are prime? We don't want to check all the numbers up to 73, as an example. For a number like 72, there are many prime factors: $72 = 2 \times 36$ and $72 = 3 \times 24$. Whenever there is a factor, we have one smaller and one larger factor (except for perfect squares).

So for perfect squares, we can get some ideas about prime numbers and where to check.

Prime	2	3	5	7	11	13	17	19
Square	4	9	25	49	121	169	289	361

So if we wanted to check to see if 73 was prime, this means we find where 73 would be in the "squares" list… and stop at the prime below it. $73 < 121$, so out of all the primes, the only ones to check are 2, 3, 5, and 7. As you can see, we have much less to check with this system as we wouldn't need to check all of the numbers up to 73: 1, 2, 3, 4, …, 72.

<u>Example:</u> Check to see if 133 is prime.

Find where 133 is located in the table above – notice that it is between 121 and 169. This means we need to check the primes up to 11 only (anything else would have a smaller factor).

Grab your calculator and check with division. If it goes in evenly, then 133 is not prime:

- $133 \div 2 = 66.5$ (fail)
- $133 \div 3 = 43.\overline{3}$ (fail)
- $133 \div 5 = 26.6$ (fail)
- $133 \div 7 = 19$ (works!)

Because $133 = 7 \times 19$, we know that 133 is not prime.

<u>NOTE:</u> Since we don't always have that table, we could always go backwards and find the square root on our calculator: $\sqrt{133} \approx 11.53$, so we would check primes up to 11 (which are 2, 3, 5, 7, and 11).

🖩 **<u>EXPLORE!</u>** Try a few of this shortcut but please use your calculator for the square root values. Find the primes we would need to check to determine if the number is prime.

	Number to check	**What primes to check**	**Prime or Composite**
A) **	67		Prime Composite
B)	91		Prime Composite
C)	41		Prime Composite
D)	153		Prime Composite

To help us factor more quickly, we'll need some tips that will help us see some factors quickly. The first few primes are 2, 3, 5, 7, and 11, so we'll give tips for a few of these. In general, for larger numbers, the calculator will be quite helpful! In fact, to check for factors, you don't need to test all the numbers up to a particular number – you only need to check up to the square root.

As you can see, it will be very helpful to be able to know if numbers are factorable by certain primes quickly. So here's some short-cut tips for some smaller primes.

Short cut for 2: Numbers that are multiples of 2, all end in even numbers. Think about some multiples of 2: 4, 10, 26, 38, 42… So if you look at a number and just consider the last digit, if the last digit is even (0, 2, 4, 6, 8), then the entire number has 2 as a factor.

EXPLORE! Determine if 2 is a factor of the following numbers – if it is, rewrite using 2 as a factor:

A) 47

C) 900

B) 62

D) 39

Short cut for 5: Numbers that are multiples of 5, all have a particular pattern too. Think about some multiples of 5: 15, 30, 40, 65, 75… So if you look at a number and just consider the last digit, if the last digit is 0 or 5, then the entire number has 5 as a factor.

EXPLORE! Determine if 5 is a factor of the following numbers – if it is, rewrite using 5 as a factor:

A) 47

C) 900

B) 60

D) 51

Short cut for 3: Numbers that are multiples of 3, also have a particular pattern, but it is a bit harder to see. The key for 3 is that 3 must be a factor of the sum of the digits. To check something like 18, we would add 1 + 8 = 9, and since 3 is a factor of 9, then 3 must be a factor of 18.

EXPLORE! Determine if 3 is a factor of the following numbers – if it is, rewrite using 3 as a factor:

A) 47

C) 72

B) 87

D) 319

🖩 **EXPLORE!** For the rest of the primes, you'll just use your calculator! So now, let's practice some factoring. For each number below, factor into prime numbers or state that it is prime. Remember our tools: (1) short-cut tips for small primes, (2) square table to know what primes to check, and (3) calculator to check.

	Number to check	Prime or Composite	Prime Factorization
A) **	47	Prime Composite	
B) **	60	Prime Composite	
C)	111	Prime Composite	
D)	31	Prime Composite	
E)	125	Prime Composite	
F)	84	Prime Composite	
G)	1	Prime Composite	
H)	91	Prime Composite	
I)	211	Prime Composite	

Prime	2	3	5	7	11	13	17	19
Square	4	9	25	49	121	169	289	361

4.4: Using FLOF to Simplify Fractions

The factoring knowledge from the previous section can help us make fractions look easier. We already learned that FLOF can help us rewrite fractions to build up and have larger numerators and denominators… but can we use this same knowledge to go the other direction?

Review questions: What is FLOF? Show how to use it in rewriting $\frac{3}{8}$ in two different ways.

Factoring is extremely helpful with fractions, as we'll be able to undo the FLOF! $\frac{8}{14}$ is a fraction that could be reduced to have the same value but smaller numerator/denominator. We'll factor both numerator and denominator into primes. Then, instead of multiplying with a common factor, we can divide out a common factor. $2 \div 2 = 1$, so we can cross out one of the twos in the numerator and denominator and replace them with 1.

Example: $\frac{8}{14} = \frac{2\cdot 2\cdot 2}{2\cdot 7} = \frac{\overset{1}{\cancel{2}}\cdot 2\cdot 2}{\underset{1}{\cancel{2}}\cdot 7} = \frac{2\cdot 2}{7} = \frac{4}{7}$

Example #2: Let's do another example with $\frac{18}{24}$: $\frac{18}{24} = \frac{2\cdot 3\cdot 3}{2\cdot 2\cdot 2\cdot 3} = \frac{\overset{1}{\cancel{2}}\cdot \overset{1}{\cancel{3}}\cdot 3}{2\cdot 2\cdot \underset{1}{\cancel{2}}\cdot \underset{1}{\cancel{3}}} = \frac{3}{2\cdot 2} = \frac{3}{4}$.

You could also find factors that are not prime – it is up to you: $\frac{18}{24} = \frac{3\cdot 6}{4\cdot 6} = \frac{3\cdot \overset{1}{\cancel{6}}}{4\cdot \underset{1}{\cancel{6}}} = \frac{3}{4}$

When the only common factor between numerator and denominator is 1, we say that the fraction is in **simplest form**. Any use of FLOF from that point would make the numbers larger.

EXPLORE! Practice this new skill by simplifying the following fractions. When indicated by the symbol, use the calculator to simplify. Some of these may already be simplified if they share no common factors.

A) ** $\frac{27}{60}$

C) $\frac{150}{350}$

E) $\frac{27}{40}$

B) $\frac{30}{55}$

D) $\frac{16}{24}$

F) 🖩 $\frac{3,024}{6,120}$

4.5: Multiplication of Fractions

Multiplication of fractions is not that much different than other multiplication, let's check to see how it matches up with the other methods. What would be the product of 4 and $\frac{3}{7}$? We'll write it out with repeated addition to check it out and find the result.

$$4\times\left(\frac{3}{7}\right)=\left(\frac{3}{7}\right)+\left(\frac{3}{7}\right)+\left(\frac{3}{7}\right)+\left(\frac{3}{7}\right)=\frac{12}{7}$$

EXPLORE! Now you try a few – rewrite with addition and then find the result:

A) $8\times\left(\frac{2}{3}\right)$

B) $\left(\frac{5}{9}\right)\times 4$

This does give a good idea about how to multiply these quickly: $a\times\left(\frac{b}{c}\right)=\frac{a\times b}{c}$. Let's try a few of these problems without writing them out as repeated addition.

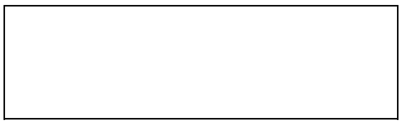
EXPLORE! Simplify the product if you can.

A) $8\times\left(\frac{5}{6}\right)$

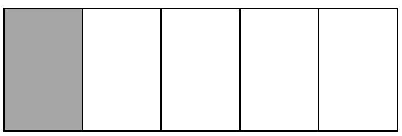
B) $\left(\frac{2}{7}\right)\times 21$

To determine what to do when we multiply a fraction by a fraction, like $\frac{2}{3}\times\frac{1}{5}$, let's draw a picture.

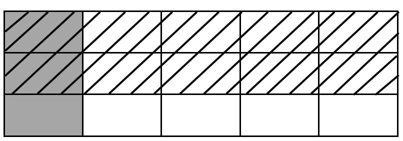
This box shape will represent one whole object.



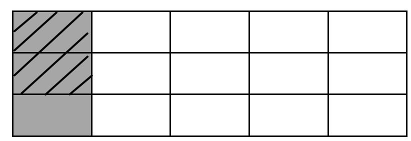
So if we wanted to find $\frac{1}{5}$ of this shape, we could break it into 5 chunks (equal sized) and then shade one of them. That would represent $1\times\frac{1}{5}$ or $\frac{1}{5}\times 1$.



So how to find $\frac{2}{3}\times\frac{1}{5}$? Start with the same $\frac{1}{5}$, but this time, break the shape into thirds as well – this time horizontally. Shade two of those three pieces with a different shade.



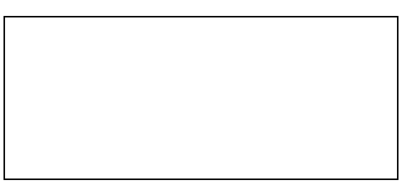
So out of our $\frac{1}{5}$ we have the $\frac{2}{3}$ of it that we want.



So the result of $\frac{2}{3}\times\frac{1}{5}$ will be the pieces that are shaded both ways. For our problem, there are two boxes shaded both ways out of the 15 equal sized pieces making one whole object; this shows $\frac{2}{3}\times\frac{1}{5}=\frac{2}{15}$.

- Since we broke the object into three equal pieces in one direction and five equal pieces in the other, we can see there will be 15 pieces total… coming from the product of 3 and 5.
- And the two boxes shaded both ways are formed by two in one direction and one in another.
- This gives the idea that $\frac{2}{3}\times\frac{1}{5}=\frac{2\times 1}{3\times 5}$.

Create a picture with appropriate shading that will represent $\frac{2}{5}\times\frac{3}{4}$… use the box to help.



Based on the picture, what is the result to $\frac{2}{5}\times\frac{3}{4}$?

<u>NOTE:</u> *This shows why it works. However, we will not ask you to perform this on a quiz or test, it is for your information only!* ☺

For multiplication, this process does generalize as we could create pictures for any product of fractions that would be similar to these.

Definition: For any fractions $\frac{a}{b}$ and $\frac{c}{d}$, the product of the two is $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$.

EXPLORE! Use the definition of fraction multiplication to find the following (simplify the result):

A) ** $\frac{40}{49} \times \frac{7}{20}$

B) $\frac{3}{7} \times \frac{4}{5}$

C) $\frac{3}{5} \times \left(-\frac{20}{21}\right)$

D) $\left(-\frac{36}{81}\right) \times \left(-\frac{9}{20}\right)$

One thing to notice about simplifying fractions is that the simplifying is fairly challenging when the numbers are very large. When possible, it would be nice to simplify before you multiply. Remember that simplifying fractions was based on FLOF – and common factors that are divided out. Here's why Dr. Phil tells us to *"Simplify before you multiply."*

Let's consider $\frac{40}{49} \times \frac{7}{20}$ from above. If we multiply the numbers straight across, $\frac{40}{49} \times \frac{7}{20} = \frac{40 \times 7}{49 \times 20} = \frac{280}{980}$, and then we'd have to simplify this fairly large fraction.

Let's try a new simplifying technique, a technique that works with multiplication (because that's where factors come from).

$$\frac{40}{49} \times \frac{7}{20} = \frac{40 \times 7}{49 \times 20} = \frac{\overset{2}{\cancel{40}} \times 7}{49 \times \underset{1}{\cancel{20}}} = \frac{2 \times \overset{1}{\cancel{7}}}{\underset{7}{\cancel{49}} \times 1} = \frac{2}{7}$$

When we see common factors, we could simplify them immediately. For example, with the 40 and the 20, we could see a common factor of 20, and divide that out of both. Then if you see the 7 and the 49, each has a factor of 7 and we could divide that out as well. By simplifying first, we could get the result much quicker than if we multiplied first and then simplified.

Remember Dr. Phil and: *“Simplify before you multiply.”*

EXPLORE! Try a few of these multiplications – and please, for your own sake, “Simplify before you multiply!”

A) ** $\frac{21}{40} \times \frac{10}{14}$

D) $\frac{7{,}200}{3{,}600} \times \frac{25}{75}$

B) ** $\frac{81}{16} \times \frac{32}{18} \times \frac{4}{5}$

E) $\frac{9}{11} \times \frac{55}{12}$

C) $\frac{15}{26} \times \frac{4}{25} \times \frac{13}{3}$

F) $\frac{40}{35} \times \frac{15}{20} \times \frac{700}{600} \times \frac{9}{12}$

Now we want to look at the properties and make sure we see how the properties can be used correctly, so we'll ask whether this list has properties used correctly or incorrectly.

	Property Name	Using it like this…	Is…
A) **	Distributive Property	$3(a+9)=3a+9$	Incorrect Correct
B) **	Additive Identity	$5+3+0=5+3$	Incorrect Correct
C)	Additive Inverse	$16\times 0=0$	Incorrect Correct
D)	Distributive Property	$5(a-9)=5a-45$	Incorrect Correct
E)	Commutative Property	$45\times 19=19\times 45$	Incorrect Correct
F)	Associative Property	$5\times(3\times 7)=(5\times 3)\times 7$	Incorrect Correct
G)	Commutative Property	$5\times(3\times 7)=(3\times 7)\times 5$	Incorrect Correct
H)	Distributive Property	$-2(5a-3)=-10a-6$	Incorrect Correct
I)	Multiplicative Identity	$5\times 1=5$	Incorrect Correct

<u>4.6: Multiplication of Decimals</u>

Multiplication of decimals is linked nicely to fractions, because every fraction can be written as a decimal – and many decimals can be written as fractions.

How could we multiply something like 3.2×4.1?

Before we do the multiplication, let's get a feel for the result. Since 3.2 is close to 3 and 4.1 is close to 4, the result should be close to 12. Because we rounded both numbers down, the exact product should be greater than 12. This method of estimating is helpful to compare with the result and check our work.

<u>Example #1:</u> In order to find the product, it may be helpful to rewrite these decimals as fractions to see the patterns. $3.2 \times 4.1 = \frac{32}{10} \times \frac{41}{10} = \frac{32 \times 41}{10 \times 10} = \frac{1{,}312}{100} = 13.12$. This is bigger than 12, so it fits!

<u>Example #2:</u> Let's practice another one to see about a pattern: 1.5×0.37.

$$1.5 \times 0.37 = \frac{15}{10} \times \frac{37}{100} = \frac{15 \times 37}{10 \times 100} = \frac{555}{1{,}000} = 0.555.$$

There's definitely a connection here and perhaps a way to make this process even more efficient! Example 1 had one decimal place in each factor and had two decimal places in the product. Example 2 had one decimal place in one factor and two in another, with three total in the product. Notice the denominators as well. Remember a few sections ago when we did $10 \times 100 = 1{,}000$ and other products of 10. The end result was the sum of the number of zeros, so let's make a process linked to that!

<u>**EXPLORE!**</u> Using our patterns and knowledge of multiplication, predict how many decimal places the product will have. Once you have your prediction down, use the calculator to determine the resulting product and see if your prediction is correct!

	Problem	Number of predicted decimal places	Answer from calculator 🖩
A) **	15.3×7.002		
B)	1.35×70.17		
C)	0.98×0.003		
D)	0.98×0.05		

E) Are there any problems with our predictions?

Create a rule about how to multiply decimals (be sure to include the number of decimal places):

EXPLORE! Put your rule to the test – find the product and show your work. Perform all multiplication by hand unless the calculator symbol is included.

A) 1.04×0.5

C) 0.011×0.04

B) $0.7 \times (-2.3)$

D) 🖩 $0.987 \times (-0.0147)$

When we work with addition and subtraction using a calculator, it's very good to have the number sense to **estimate** (or approximate) the value. Estimating is a way to get close to the answer quickly. For estimating, we don't care about what the actual value is, but just about getting close to it.

	Estimating the value of …	Is closest to…
A) **	4,000 × 1.1	400 4,000 40,000 400,000
B) **	$7 \times \frac{1}{5}$	7 $\frac{1}{5}$ 0 1
C)	4,000 × 11	400 4,000 40,000 400,000
D)	4,000 × 0.11	400 4,000 40,000 400,000
E)	$\frac{1}{8} \times \frac{1}{5}$	8 40 0 1
F)	52 × 39	200 2,000 20,000 200,000
G)	5.2 × 0.39	0.002 0.02 0.2 2 20

4.7: Number Sense and Multiplication

When we multiply numbers together, many people think that the size goes up. And in one manner of thinking, it does: 7×5 will be bigger than 7 and bigger than 5.

What about 7×0.5? When we multiply those out, $7 \times 0.5 = 3.5$ which is larger than 0.5 and smaller than 7. The size of the product depends on the factors, but the key idea to understand relates to 1 and 0. The definition of multiplication shows that $a \times 0 = 0$ and the multiplicative identity property shows $a \times 1 = a$.

Think back to what we've learned about the sign of the product?

A) pos × pos =

B) pos × neg =

C) neg × pos =

D) neg × neg =

Once the sign is determined, we can focus on the sign of the product by multiplying the absolute values (or sizes). But what happens to the size (absolute value) of the product?

Interactive Examples: Circle whether the product is greater or less than each factor.

	If the problem is…	Then the product will be …	
A) **	4×0.5	Greater Less than 4	Greater Less than 0.5
B)	$5 \times \left(1\frac{3}{7}\right)$	Greater Less than 5	Greater Less than $1\frac{3}{7}$
C)	6×0.375	Greater Less than 6	Greater Less than 0.375
D)	$7 \times \frac{12}{5}$	Greater Less than 7	Greater Less than $\frac{12}{5}$

To summarize the results, we can make a list of what we saw – remember these are describing the "size" and not focusing on the "sign":

	If the size of B is…	Then A × B will be …			
E) **	0	Greater than A	Less than A	Equal to A	Equal to 0
F)	Between 0 and 1	Greater than A	Less than A	Equal to A	Equal to 0
G)	1	Greater than A	Less than A	Equal to A	Equal to 0
H)	Greater than 1	Greater than A	Less than A	Equal to A	Equal to 0

EXPLORE! Determine the size and sign of the product without performing actual multiplication.

	Product	Sign	Size (absolute value) is…
A) **	$\frac{2}{3}\times(-5)$	Pos Neg Zero	Greater than 5 Less than 5
B) **	$\frac{2}{3}\times\frac{2}{3}$	Pos Neg Zero	Greater than $\frac{2}{3}$ Less than $\frac{2}{3}$
C)	$(-11)\times\left(-\frac{7}{9}\right)$	Pos Neg Zero	Greater than $\frac{7}{9}$ Less than $\frac{7}{9}$
D)	$(-11)\times(0.05)$	Pos Neg Zero	Greater than 11 Less than 11
E)	$(6.92)\times\left(-\frac{7}{9}\right)$	Pos Neg Zero	Greater than 6.92 Less than 6.92
F)	$\left(-\frac{7}{9}\right)\times(-0.92)$	Pos Neg Zero	Greater than 0.92 Less than 0.92
G)	$(-6.92)\times(12.5)$	Pos Neg Zero	Greater than 6.92 Less than 6.92
H)	$(-6.92)\times(0.15)$	Pos Neg Zero	Greater than 0.15 Less than 0.15

4.8: Repeated Multiplication (Exponents)

Early in this course, we noticed that repeated addition could be simplified: $5+5+5+5+5+5=6\times 5$. Is there a way to simplify repeated multiplication like $2\times 2\times 2\times 2$? Of course, we could multiply this out and find that $2\times 2\times 2\times 2=4\times 2\times 2=8\times 2=16$. But it would be very hard to write out the product of 17 twos… or 35 twos. For this, we introduce some new mathematical notation called exponents (or powers).

$2\times 2\times 2\times 2=2^4$. This notation is our way of indicating repeated multiplication in a more efficient way. Let's have you try a few – rewrite the repeated multiplication using exponents.

A) $5\times 5\times 5\times 5\times 5\times 5\times 5\times 5=$

B) $4\times 4\times 4\times 4\times 4\times 4\times 4\times 4\times 4\times 4\times 4\times 4=$

Definition: For numbers *n* and *a*, we write *a* to the power of *n* as a^n, where $a^n=\underbrace{a\times a\times a\times\ldots\times a}_{n\text{ times}}$.

We call *a* the **base** and *n* the **power** or **exponent**. For this definition, we'll keep $n>0$. Later we may discover what happens when the exponent is zero or negative.

EXPLORE! Try simplifying some more repeated multiplication with a few that have different bases – rewrite the repeated multiplication using exponents. Then calculate the value of the result.

A) ** $0.3\times 0.3\times 0.3$

B) ** $-8\times 8\times 8$

C) $\left(-\frac{7}{9}\right)\times\left(-\frac{7}{9}\right)$

D) $(-1)\times(-1)\times(-1)\times(-1)\times(-1)$

Make sure you are careful with the use of negatives with exponents.

EXPLORE! Try some where we go backwards – write these out then calculate the value of the result.

A) ** $(-2)^3$

B) ** $-(-3)^2$

C) $-(1)^3$

D) $-\left(-\frac{2}{3}\right)^2$

E) $-(-2)^3$

F) $-(5)^2$

EXPLORE! Use your calculator to find the following values. Find the exponent button on your calculator; it will usually look like one of the following: $\boxed{y^x}$ $\boxed{x^y}$ $\boxed{\wedge}$

A) $7^6 =$

C) $\left(\sqrt{5}\right)^4 =$

E) $(-3)^2 =$

B) $(3.87)^3 =$

D) $\left(-\frac{4}{5}\right)^4 =$

F) $-3^2 =$

What happened with the last two: $(-3)^2$ and -3^2? Why did these give a different result when calculated? Try to explain the difference between the two using the word "exponent" and "base."

EXPLORE! Without a calculator, based on the explanation above, determine the following values:

A) $(-4)^2$

C) -4^2

E) $(-2)^3$

B) -8^2

D) $(-9)^2$

F) -1^3

NOTE: *Without exponents, it would be extremely tedious to write out the result as repeated addition, but we could. Look how long it would take for us to rewrite* 3^3*:*

$3\times(3\times3)=$
$(3\times3)+(3\times3)+(3\times3)=$
$(3+3+3)+(3+3+3)+(3+3+3)=$
27

Long story short: exponents are our friends!

4.9: Multiplication with variables

Multiplication with variables requires the ability to work with exponents. For example, if we wanted to multiply $x \cdot x \cdot x$, we could rewrite it using exponents. $x \cdot x \cdot x = x^3$.

But what about $x^3 \cdot x^5$? We could write it efficiently: $x^3 \cdot x^5 = (x \cdot x \cdot x) \cdot (x \cdot x \cdot x \cdot x \cdot x) = x^8$

EXPLORE! Simplify the following.

a. ** $x^9 \cdot x^4 =$

b. $x^{17} \cdot x^{25} =$

c. $x^5 \cdot x^{14} \cdot x^{24} =$

d. $x^9 \cdot x^{-4} =$

e. $x^9 \cdot y^{11} \cdot x^3 \cdot y^{45} =$

Create a rule that demonstrates this property: $x^A \cdot x^B =$

Think about the last two parts above: when simplifying the exponents using our rule, does the base have to be the same?

Now let's consider $(x^3)^4$? What does this expression represent, how could we simplify it and write it more efficiently – without parenthesis?

Remember the definition of exponents as repeated multiplication: $(x^3)^4 = x^3 \cdot x^3 \cdot x^3 \cdot x^3 = x^{12}$.

EXPLORE! Simplify the following.

A) ** $(x^9)^4 =$

B) $(x^{17})^2 =$

C) $(x^{11})^3 \cdot (x^8)^2 =$

D) $(x^{11})^3 \cdot (x^8)^{-2} =$

Create a rule that demonstrates this property: $(x^A)^B =$

Think about the last two parts above: when simplifying the exponents using our rule, does the sign on the exponent change the rule?

Now let's consider $(4x)^2$? What does this expression represent, how could we simplify it and write it without parenthesis?

Remember the definition of exponents as repeated multiplication, and using the commutative and associative properties: $(4x)^2 = 4x \cdot 4x = 4 \cdot 4 \cdot x \cdot x = (4)^2 \cdot (x)^2$.

<u>EXPLORE!</u> Simplify the following.

A) ** $(2x)^3 =$

C) $(-5x^{23})^2 =$

B) $(7x^5)^2 =$

D) $(-x^4y^5)^7 =$

Create a rule that demonstrates this property: $(xy)^A =$

<u>EXPLORE!</u> Let's put all the properties together and try a few.

- $x^A \cdot x^B = x^{A+B}$
- $(x^A)^B = x^{AB}$
- $(xy)^A = x^A y^A$

A) ** $(x^3y^2)^7 \cdot x^5 =$

B) $(x^{-3}y^{-2})^{-5} \cdot x^{16} =$

The formula $x^A \cdot x^B = x^{A+B}$ is a chance to connect our previous knowledge to this new concept. Think about how you would find the result of $10^2 \cdot 10^3 =$

EXPLORE! Find the following values based on our properties.

A) ** $10^4 \cdot 10^3$

C) $10^3 \cdot 10^3$

B) $10^2 \cdot 10^5$

D) $10^4 \cdot 10^1$

When we earlier multiplied groups of ten, we saw a slightly different pattern.

EXPLORE! Practice a few below – multiply using place value numbers:

A) ** $100 \times 100 =$

C) $10{,}000 \times 100 =$

B) $1{,}000 \times 100 =$

D) $10{,}000 \times 1{,}000 =$

Explain how "counting the 0's and adding how many there are" relates to the rule for exponents.

This is a good place to summarize some key thoughts about operations and properties. With addition, the key number is 0 and with multiplication (or exponents), the key number is 1.

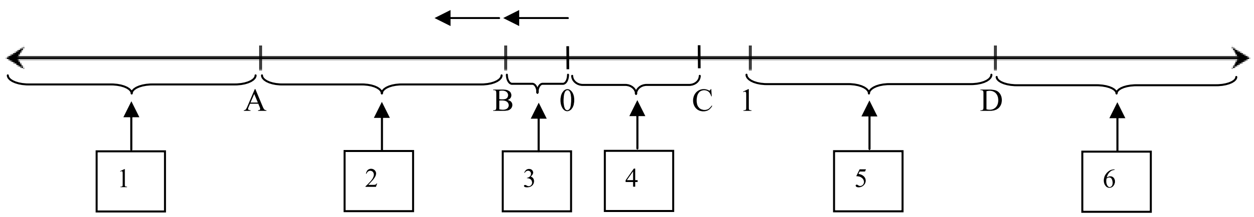
	If Y is…	**Then X + Y is…**	**Verbal Explanation**
A)	Positive	Greater than X	If we add positives, the result gets bigger.
B)	Negative	Less than X	If we add negatives, the result gets smaller.
C)	0	Equal to X	Add 0 and get what you started with.
	If Y is…	**Then X – Y is…**	**Verbal Explanation**
D)	Positive	Less than X	If we subtract positives, the result gets smaller.
E)	Negative	Greater than X	If we subtract negatives, the result gets bigger.
F)	0	Equal to X	Subtract 0 and get what you started with.
	If the size of Y is…	**Then the size of X × Y is…**	**Verbal Explanation**
G)	Greater than 1	Greater than X	More than 1 of something makes it larger.
H)	Equal to 1	Equal to X	Multiply by 1 and get what you started with.
I)	Between 0 and 1	Smaller than X	A part of something makes it smaller.
J)	Equal to 0	Equal to 0	Multiply by 0 and you'll get 0.

4.10: Multiplication on a Number Line

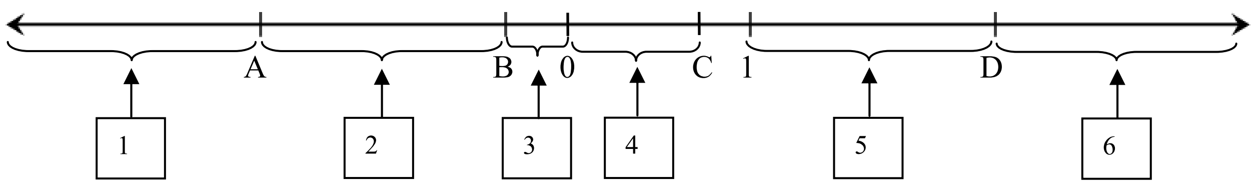
In Unit 3, we saw that subtraction can be rewritten as addition, and in Unit 4 we learned that multiplication can be thought of as repeated addition. Since 2A = A + A, let's venture back to the number line! The key pieces with multiplication are the signs (which we'll do first) and the sizes (which we'll do second).

Example: Find the region that best approximates 2B.

Method 1: To find the value, we can rewrite 2B as B + B, so we will draw in B with an arrow. So this ends up in region 2.



Method 2: Look at size and sign. First, with 2B, we have two numbers. What are their signs? One is positive and one is negative. So the product must be negative (only regions 1, 2, and 3 are possible). Now check the sizes of the numbers. Since the size of 2 is greater than 1, and the size of B is less than 1, the result must have a larger size than B and a smaller size than 2. Only region 2 is possible!



This second method is probably the easiest to use. Let's try some!

EXPLORE! Practice more with the number lines.

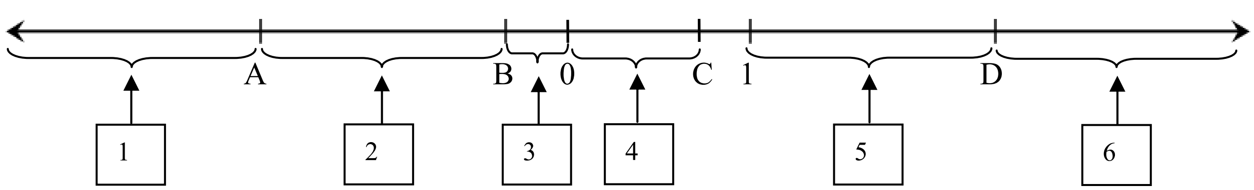
A) ** Find the region that best approximates C × D.

B) Find the region that best approximates A × C.

C) ** Find the region that best approximates 2B + D.

D) Find the region that best approximates D – 2B.

E) Find the region that best approximates C^2.

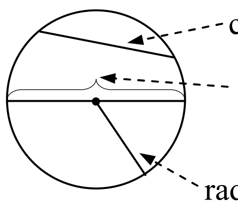


4.11: 2-Dimensional Geometry and Multiplication

When showing fraction multiplication, we saw the construction of a box split into pieces. Multiplication can be defined in terms of the same type of box.

In a previous unit, we saw that the perimeter could be represented as the sum of all the lengths of a geometric figure. However, that definition is related to shapes with straight line sides since the length of a curved shape is much harder to come up with.

For a circle, the terms often related are chord, center, circumference, diameter, and radius. Every circle has a center point, and the fixed distance from the **center** (or origin) to a point on the edge is called the **radius**. A **chord** of the circle connects two points on the circle, while the **diameter** is a chord that passes through the center. Since a diameter passes through the center, you could think of the diameter as being composed of two radii. The **circumference** is the perimeter of a circle.



People began measuring aspects of circles and comparing them. They noticed a pattern, like we might in Unit 1 – as the diameter increased, the circumference increased as well. The sizes of the different measurements were compared, and it became clear that the circumference was always a little more than three diameters. This ratio of circumference to diameter never changed, no matter how big the circles were, and since it never changed, this mathematical constant took on its own name, pi (π).

$$\pi = \frac{C}{d} = \frac{\text{Circumference}}{\text{diameter}}$$

Mathematicians have now calculated π to trillions of decimal places, but all of those calculations are just approximations. Using your calculator, find the button for π and write out as many decimal places as you can in the space below.

$\pi \approx$

With this value, we could rewrite the ratio $\pi = \frac{C}{d}$ as $C = \pi \cdot d$, indicating that there are "π" lengths of the diameter that will make the circumference. This last equation gives a formula in order to calculate the circumference if we know the diameter.

Example: Find the exact value and an approximation of the circumference of a circle with a diameter of 8 inches.

Since $C = \pi \cdot d$ then $C = \pi \cdot (8\text{ inches}) = \underbrace{8\pi\text{ inches}}_{\text{Exact Answer}} \approx \underbrace{25.13274\text{ inches}}_{\text{Approximation}}$.

Be careful – if you use the π button at any time with your calculator, the result becomes an approximation and is no longer exact.

<u>EXPLORE!</u> Compute the circumference of the circle given the information below. State the exact value and then use the calculator to approximate the value to 4 decimal places.

A) ** diameter is 5 feet

B) diameter is 8.9 centimeters

C) radius is 3 inches

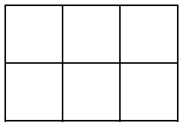
D) radius is 28 centimeters

<u>For Love of the Math (a piece of pi):</u> *For thousands of years, humans around the world have looked for better approximations of π with excellent results.*

- *Babylonians (roughly 17 centuries BCE) – used* $\frac{25}{8}$.
- *Egyptian (about 1600 BCE) – in the Rhind Mathematical Papyrus used* $\frac{256}{81}$.
- *Indian (4th century BCE) – Astronomy calculations by Brahmana used* $\frac{339}{108}$.
- *Greek (3rd century BCE) – Archimedes proved that pi was between* $\frac{223}{71}$ *and* $\frac{22}{7}$ *using lengths of 96-gons.*
- *Greek (2nd century CE) – Ptolemy used* $\frac{377}{120}$.
- *Chinese (around 250 – 500 CE) – Liu Hui used* $\frac{3{,}927}{1{,}250}$*, and Zu Chongzhi used* $\frac{22}{7}$ *and* $\frac{355}{113}$.
- *These approximations were used repeatedly until more advanced techniques came up; the longest hand calculation was William Shanks at over 500 digits (1873).*
- *Nowadays, approximations of π are done with formulas and computers.*

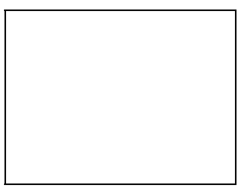
The idea of circumference can be written with diameter, as $C = \pi \cdot d$, but since one diameter can be replaced by two radii, we could also write $C = \pi \cdot (2r)$ or more commonly, as $C = 2 \cdot \pi \cdot r$.

The circumference of a circle is not the only geometry topic where multiplication is used. Think of a box that is 3 feet long by 2 feet wide. When you draw the box, you'll see something like:



Each small box would be 1 foot by 1 foot, which we have defined as 1 square foot = 1 ft^2. How many total square feet are in the box total?

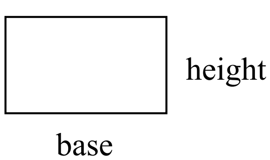
Interactive example: Use the box below to create a shape that is 3 by 4 and determine the number of 1 by 1 squares inside.

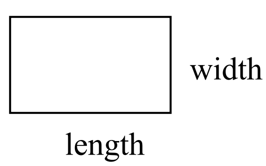


This is the basis for how we define **area**, the two-dimensional concept of filling space. While the perimeter describes how far it is around a shape, the space filled by the shape is the area. With many different shapes, multiplication is the key to almost all area formulas!

In order to see how multiplication comes up, we'll take a look at some basic shapes and determine how to come up with a formula to calculate the area more quickly.

A **rectangle** has four sides and all right angles, while a **square** is a rectangle with all sides the same. Sometimes we use length and width – where the length is usually longer than the width. However, it may be better to define the sides as base and height. Both labels are shown below.





Stacking 1 by 1 squares inside and counting them would give us the area. Each row length is the length of the base, and by stacking these rows over and over again, we repeatedly add the value. Remind you of anything? Repeated addition, which is multiplication!

The area of a rectangle or square can be computed using $A = bh$ or $A = lw$.

What about a shape that is not a rectangle or square? We will stick to four sided figures for the next portion, but this time we'll allow the angles to not be right angles. However, we'll still have two pairs of side lengths creating a **parallelogram**.

For this shape, there are still sides, but what about the height? How high is this parallelogram? Should we measure the height by the slanted side, or use something else?

Interactive Example: Draw in the height on the shape above and describe what makes it the height:

In order to determine the area of this shape, we could perform a few adjustments and the shape will become like one we've already seen! Draw in a height (Figure 1), and notice how it creates a triangle. Take that triangle and cut it off, and then slide it directly across the parallelogram (Figure 2). Once you sketch the resulting figure, you'll have a nice rectangle (Figure 3).

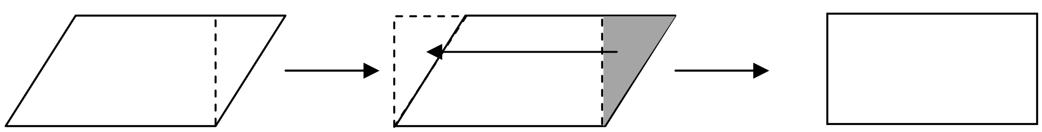
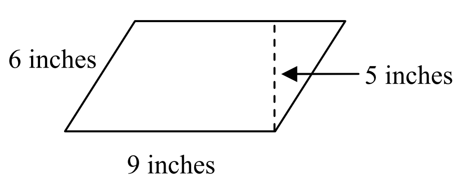


Figure 1 Figure 2 Figure 3

Since the formula for a rectangle was $A = bh$, this is exactly the same formula for parallelograms. But you have to be a little careful with this – the side length is not the same as the height!

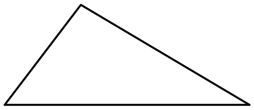
Example: Find the perimeter **and** area of the following shape.



The perimeter is found by the outer edges (6 inches and 9 inches), so P = 6 + 9 + 6 + 9 = 30 inches. The area is found by base (9 inches) times height (5 inches): $A = (9\text{ in})(5\text{ in}) = 45\text{ in}^2$.

Mathematicians love this concept: taking some new idea, new shape, or new number, and somehow turning it into what was already known. We used this idea when turning a parallelogram into a rectangle, so let's try another extension like this.

A **triangle** is a shape with three sides, and we have previously found the perimeter. But the area is a different story. There's a triangle below – label the base and the height.



Since this shape doesn't look the same as the previous, it's hard to see the connection or how we can use our previous knowledge. But this time, we'll start with the triangle (Figure 1) make a copy of the shape and then rotate it (Figure 2). By drawing the outside of the new figure, we can observe a shape we've seen before – a parallelogram (Figure 3)!

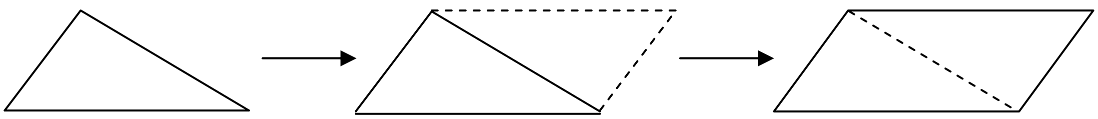


Figure 1 Figure 2 Figure 3

The area of the parallelogram was $A = bh$, so we could use that formula to compute the area of the parallelogram here. However, if we wanted to know the area of the triangle, the parallelogram formula will not return the correct value.

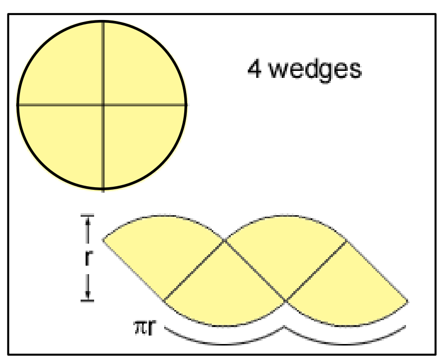
How many triangles are in the parallelogram?

Let's tweak the parallelogram formula to make a triangle formula: $A =$

Again, it's important to see that the height may not be a side length.

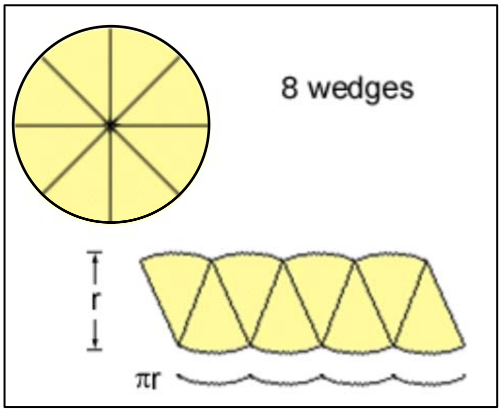
We will use one more example of this, and it may be the most interesting yet. Since a circle is curved, it does make it a challenge to turn into a shape we've already seen; but we shall rise to the challenge.

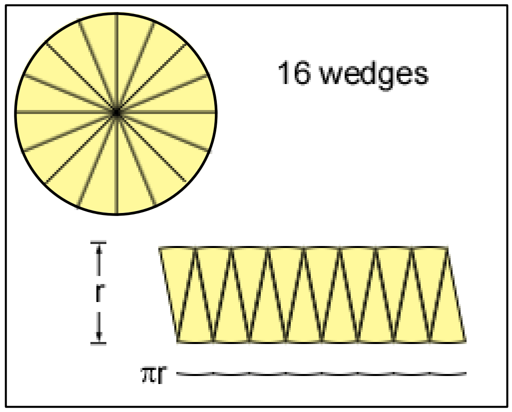
With a circle, we already know that the circumference is found by $C = 2 \cdot \pi \cdot r$. How to create the area then? Let's start by breaking up a circle into pieces. We'll start with 4 pieces, and rearrange the pieces similar to the triangle. However, this still doesn't look like much.



Next, we'll cut it into more pieces and see what happens. Notice that the bottom ridges are formed by half the circumference, or $\pi \cdot r$.

With the next two pictures, we show that as the number of wedges we cut increases, the rearranged shape starts to look more and more like something we do know about, a parallelogram!





Pictures found on: http://britton.disted.camosun.bc.ca/areacirc/areacirc.htm

What is the area formula for a parallelogram?

In the picture from 16 wedges, what is the base? _______________ What is the height? ________

Use this information to create an area formula for a circle: $A = \pi \cdot r^2$

EXPLORE! Find the areas of the shapes using the formulas we have found to this point. Draw a sketch of the shape as well. Find the exact value and an approximation to 4 decimal places (if needed).

A) ** Circle with radius of 7 inches (round to 2 decimal places).

B) Circle with diameter of 13 cm (round to 2 decimal places).

C) Rectangle with height of 8 feet and base of 12 feet.

D) Parallelogram with base of 8 feet and height of 6 feet.

E) ** Triangle with base of 2.5 inches and height of 3.8 inches.

F) Square with lengths of $\frac{5}{7}$ m.

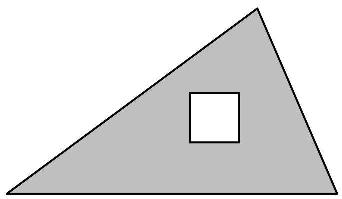
Geometry Applications - Extensions

For the following problems, find the area of the shaded region when there is a hole. This follows other parts of the course which is: find the Big region (has too much) then subtract the Small amount.

Area formulas might help here:

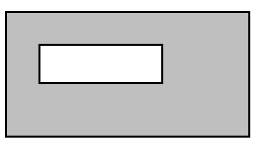
- Rectangle (square/parallelogram): $A = bh$
- Triangle: $A = \frac{1}{2}bh$
- Circle: $A = \pi \cdot r^2$

A) The shape has a triangle with height of 10 inches and base of 14 inches, and the square has 2 inch sides – find the area in square inches.

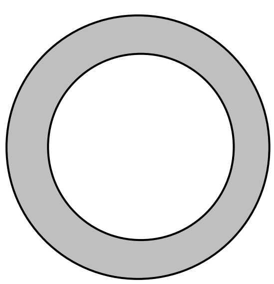


Here the area of the triangle is 70 square inches $(A = \frac{1}{2}bh)$ *and the area of the square is 4 square inches. So when we want to find the shaded region, we'll take the big shape area and subtract the small shape area. Our final area is 66 square inches.*

B) A brownie pan had a slice cut out. If the pan was 9 inches by 11 inches, and the slice was a rectangle measuring 2 inches by 5 inches, find the area in square inches.

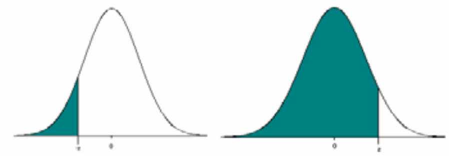


C) This is a concrete circular driveway, where the inner radius is 50 feet, and the outer radius is 65 feet. Find the area in square feet.



This "Big minus Small" concept comes up in statistics too. We'll focus on the chart/table here and seeing the connection.

When finding the amount of area under a particular curve, we use a table similar to the one below. The table finds the area from $-\infty$ to the z-value. The possible pictures look like these (first one is a z-score less than 0, and the second has a z-score greater than 0).



For this curve, half the area is below $z = 0$, which is why the area for $z = 0.00$ is 0.5000 (like 50%).

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
– 3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
– 2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
– 2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
– 2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
– 1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
– 1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
– 1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
– 0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
– 0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
– 0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
– 0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Example: Find the area under the curve and below $z = 1.25$.

To find this amount, look up in the table and find where the 1.2 row meets the 0.05 column. The area is 0.8944.

EXPLORE! Find the areas listed using the table above.

A) ** The area below $z = -1.57$.

B) ** The area between $z = 0.94$ and $z = 1.89$.

C) The area below $z = 2.13$.

D) The area below $z = 0.08$.

E) The area between $z = -2.44$ and $z = -0.60$.

F) The area between $z = -1.22$ and $z = 2.71$.

4.12: Multiplication Applications

When showing fraction multiplication, we saw the construction of a box split into pieces. Multiplication can be defined in terms of the same type of box.

Money Applications:

A) If someone earns $11.50 per hour, and works 35 hours per week, how much would they earn in one week?

B) If someone earns $10 per hour for regular pay, and "time and a half" for overtime, what is the rate of pay for overtime hours? [Overtime hours are counted for more than 40 hours per week]

C) The person earning $10 per hour works 46 hours per week, how much did they earn that week?

D) Who earns more over the course of one month? Alicia earns $18 per hour and works 40 hours per week. Luisa earns $15 per hour and works 45 hours per week.

Sports Applications:

A) In a season, a football player ran for an average of 4.7 yards per carry, and had 217 carries; how many yards did the player rush for that season?

Other Applications:

A) 12 groups of 12 is known as a **gross**. If a shipment of pencils included 36 gross and 8 dozen, how many total pencils were there?

B) If a large table holds 12 people, and small table holds 7 people, how many people can be seated at a table in a room with 13 large tables and 21 small tables?

4.13: Multiplication Summary

When dealing with multiplication, there are many ways to visualize the concepts – repeated multiplication, the product as an area, and repeated multiplication as exponents. This review pages will help to tie the concepts together.

Explain/Summarize the following addition properties:

- Commutative Property of Multiplication

- Associative Property of Multiplication

- Identity Property of Multiplication

- Distributive Property of Multiplication over Addition

	If the size of B is…	**Then A × B will be …**			
A)	0	Greater than A	Less than A	Equal to A	Equal to 0
B)	Between 0 and 1	Greater than A	Less than A	Equal to A	Equal to 0
C)	1	Greater than A	Less than A	Equal to A	Equal to 0
D)	Greater than 1	Greater than A	Less than A	Equal to A	Equal to 0

Concept Questions

When dealing with integers, the sign of the result depends on the sign of the starting numbers. The result could be always positive (P), always negative (N), or sometimes positive and sometimes negative (S). Label each of the following expressions as P, S, or N. If the answer is P or N, explain why. But if the answer is S, give one example that shows the product could be positive and one example that shows the product could be negative. Note*: some questions review previous topics as well as multiplication.*

	Expression	**Sign (circle one)**	**Examples or Explanation**
E)	pos × pos	P S N	
F)	pos × neg	P S N	
G)	neg × neg	P S N	
H)	neg × pos	P S N	