Math Fundamentals for Statistics (Math 52)

Unit 5: Division

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“The ‘How’ and ‘Whys’ Guys”

5.1: Division

In mathematics, there are really only two operations at this stage: addition and multiplication. If we remember that subtraction can be rewritten as adding the additive inverse (or opposite), then division will be fairly similar. It's really helpful to be able to define division so that we're clear on what we mean by $a \div b$. To be consistent, we'll define division in terms of multiplication – this connects the field properties to what we're doing.

Definition: The division of *a* and *b* is written $a \div b$ or $\frac{a}{b}$. $a \div b = c$ will be true if, and only if, $b \times c = a$, and the result of the division, *c*, must exist and be unique. From the equation $a \div b = c$, *a* is called the **dividend**, *b* is called the **divisor**, and *c* is called the **quotient**. If the divisor does not go in evenly, then the remaining amount added is called the **remainder** and we can write this as $b \times c + r = a$, or $a \div b = c$ R r. Note: The remainder must be less than the divisor, but never negative.

So let's put this definition to use – what is $12 \div 3$?

For many people, they can quickly see the answer. But *why* is it the answer? $12 \div 3 = 4$ because $3 \times 4 = 12$ and also there is no other number that will multiply by 3 to make 12.

EXPLORE! Let's try a few of these problems and make sure you explain why the value you find meets the definition of division.

A) ** $28 \div 4$

B) $56 \div 8$

C) $42 \div 7$

D) $38 \div 9$

E) $66 \div 10$

F) $72 \div 8$

G) $100 \div 10$

H) $1{,}000 \div 10$

I) $100{,}000 \div 100$

J) $10{,}000{,}000 \div 1{,}000$

Interactive Examples: For our purposes, we haven't seen the idea of "existence and uniqueness" come into play, but it does now. So let's try the next few and see if we can find the quotient.

A) Explain why $0 \div 7 = 0$.

B) What is the problem with $13 \div 0$?

C) What is the problem with $0 \div 0$?

EXPLORE! For the following, determine the quotient and remainder if possible. If not possible, explain why it is not possible.

A) ** $60 \div 10$

C) $0 \div 30$

B) ** $10 \div 0$

D) $0 \div 0$

5.2: Division as Subtraction (Algorithms)

Another way to think of division is as repeated subtraction. The quotient would describe how many times you'd subtract a number to get back to 0.

$12 \div 3 = 4$ because we would subtract 4 groups of 3 from 12. We could track the subtractions like this: 12 – 3 = 9, 9 – 3 = 6, 6 – 3 = 3, 3 – 3 = 0. Once you get to 0, you're done!

This description shows another reason why there is such a big problem when dividing by 0.

Example #2: Use repeated subtraction to find the quotient from $15 \div 5$.

15 – 5 = 10
10 – 5 = 5
5 – 5 = 0 (once we hit 0, we're done!)

It took 3 subtractions of 5, so $15 \div 5$ = 3.

Interactive Example: Use repeated subtraction to find the quotient from $2{,}864 \div 2$.
2,864 – 2 × 1,000 = 2,864 – 2,000 = 864. [Here we will take away large numbers of 2 at a time]
864 – 2 × 400 = 864 – 800 = 64
64 – 2 × 30 = 64 – 60 = 4
4 – 2 × 2 = 4 – 4 = 0.

Count up the groups: 1,000 + 400 + 30 + 2 = 1,432. So $2{,}864 \div 2$ = 1,432.

This subtraction (in groups) idea is the main reason for the algorithms we use to find quotients in harder problems. So for most division (except when we divide by 0), there will be a quotient. The method we use to find the quotient is usually compressed into some type of subtraction.

Algorithm #1: **SCAFFOLDING OR PARTIAL QUOTIENTS**
This algorithm works with the idea of repeated subtraction, and the algorithm keeps track of large groups of subtraction at once. Let's show how this works with the problem $1{,}894 \div 13$. The process starts by trying to find the biggest place value group of 13 that will fit in 1,894. The way we tend to write the division in the following way: $\text{divisor}\overset{\text{quotient}}{\overline{\big)\text{dividend}}}$ For the particular problem of $1894 \div 13$, we would start by writing this as $13\overline{\big)1894}$, where the quotient will be written on top of the bar.

The process is demonstrated on the next page in greater detail.

Step 1: Start with the largest place value of the dividend, in this case, thousands. Can we fit 1,000 groups of 13 in 1,894? No, we can't because $1000 \times 13 = 13{,}000$ which is much larger than 1,894. So we need fewer than 1,000 groups.

Step 2: Move to the next place value, hundreds. How many hundred groups of 13 are there in 1894? There are 100 groups of 13 in 1,894, since $100 \times 13 = 1300$. So we'll write 100 in the quotient to indicate 100 groups of 13. And we will subtract the 100 groups of 13 (1,300) under the 1,894. Check to see if there are more hundred groups of 13 that will fit in 594. Since $100 \times 13 = 1300$, and since 594 is less than 1,300, we know that we're done with the hundreds place value in the quotient.

Step 3: Move to the next place value, tens. How many ten groups of 13 are there in 594? Let's try 20 groups of 13, since $20 \times 13 = 260$ and this is less than 594. Write the 20 in the quotient to indicate the 20 groups of 13 that we will subtract, and write the 260 under the 594 to subtract. Once finished, we'll look at the result (334) and see if there are any more ten groups of 13. It looks like we could fit more ten groups, since $20 \times 13 = 260$. Write the 20 in the quotient to indicate the additional 20 groups of 13 that we will subtract, and write the 260 under the 334 to subtract. Check again, and this time, 74 will be smaller than 10 groups of 13 since $10 \times 13 = 130$. We are now done with the tens place value.

Step 4: Move to the next place value, ones. How many groups of 13 are there in 74? Let's try 5 groups of 13, since $5 \times 13 = 65$ and this is less than 74. Write the 5 in the quotient to indicate the 5 groups of 13 that we will subtract, and write the 65 under the 74 to subtract. Since 9 is less than 13, we're done with the entire problem! Then we add up the quotient portions to find the result: $100 + 20 + 20 + 5 = 145$, so the quotient is 145 with a remainder is 9. We could write this as: $1{,}894 \div 13 = 145 \text{ R } 9$

Step 1	***Step 2***	***Step 3 (part 1)***	***Step 3 (part 2)***	***Step 4***
13)1894	100 13) 1894 −1300 594	20 100 13) 1894 −1300 594 − 260 334	20 20 100 13) 1894 −1300 594 − 260 334 − 260 74	5 20 20 100 13) 1894 −1300 594 − 260 334 − 260 74 − 65 9

The nice thing about this algorithm, is that if you're not exactly sure about the quotient portion, you can still pick a number that is going to work and build up (scaffold) to the correct quotient.

Algorithm #2: **TRADITIONAL ALGORITHM – "LONG DIVISION"**

This algorithm is the one traditionally taught in US schools, and it is based on scaffolding. However, there are a few small differences. First, when you divide in, you'll need to find the largest number that divides into the quotient, but nothing larger. Then, instead of having to add each of the quotient groups, you just write the correct place value number in the quotient. Lastly, this algorithm typically doesn't write as much down for place values when subtracting from the quotient.

Step 1: Start with the largest digit in the dividend, in this case, the "1". Can we make groups of 13 out of 1? No, we can't because 1 < 13. So we write a 0 in the thousands place value of the quotient, and move to the next step. [This really represents 0 thousands of groups of 13]

Step 2: Consider the next numbers in place value order – in the hundreds place now, so there are 18 hundreds. How many groups of 13 are there in 18? We can see only 1, so write the 1 group in the hundreds place of the quotient, and multiply 1 × 13 = 13. Put the 13 under the 18 and subtract, then bring down the next digit in the dividend.

Step 3: Consider the next numbers in place value order – in the tens place now, so there are 59 tens at this stage. How many groups of 13 are there in 59? With some trial-and-error, we find there are 4 since $4\times13=52$. Write the 4 groups in the tens place of the quotient, and multiply $4\times13=52$. Put the 52 under the 59 and subtract, then bring down the next digit in the dividend.

Step 4: Consider the next numbers in place value order – in the ones place now, so there are 74 ones at this stage. How many groups of 13 are there in 74? With some trial-and-error, we find there are 5 since $5\times13=65$. Write the 5 groups in the ones place of the quotient, and multiply $5\times13=65$. Put the 65 under the 74 and subtract. Since we are at the end of the whole number values, we'll end with a remainder of 9. The leading 0 in this case doesn't change the value, so we could write the result as $1{,}894\div13=145$ R 9

Step 1	***Step 2***	***Step 3***	***Step 4***
0 13) 1 8 9 4	0 1 13) 1 8 9 4 −1 3↓ 5 9	0 1 4 13) 1 8 9 4 −1 3↓ ↓ 5 9 ↓ − 5 2 ↓ 7 4	0 1 4 5 13) 1 8 9 4 −1 3↓ ↓ 5 9 ↓ − 5 2 ↓ 7 4 − 6 5 9

NOTES:

- When working with fractions, we could write the 9 as a fraction of the divisor to get the result $145\frac{9}{13}$.
- When we get to working with decimals, we could continue dividing this out as long as necessary or desired, by putting 0 in each new decimal place to the right of the ones place.
- With both of these types, the concept of division doesn't change… just how we write it!

Algorithm #3: **SHORT DIVISION** – done with divisors that are very small and easy to work with (typically done with divisors that are single digits)

This algorithm allows us to write the division in significantly shorter space vertically, but maintain the same concept of repeated subtraction. With each place value, we perform the multiplication and then subtraction mentally – which is the main reason why this works well with small divisors and not as well with larger divisors. Let's demonstrate this algorithm with $19{,}875 \div 5$.

Step 1: Start with the largest digit in the dividend, in this case, the "1". How many times does 5 go into 1? It doesn't, because it's too small. Place a 0 over the place value and move on.

Step 2: Now consider the next portion: 19. How many times does 5 divide into 19? We know that $5 \times 3 = 15$, so the result is 3 (written in the quotient) and since 19 – 15 = 4, we write the portion left over as a smaller 4.

Step 3: Now consider the next portion: 48. How many times does 5 divide into 48? We know that $5 \times 9 = 45$, so the result is 9 (written in the quotient) and since 48 – 45 = 3, we write the portion left over as a smaller 3.

Step 4: Now consider the next portion: 37. How many times does 5 divide into 37? We know that $5 \times 7 = 35$, so the result is 7 (written in the quotient) and since 37 – 35 = 2, we write the portion left over as a smaller 2.

Step 5: Now consider the final portion: 25. How many times does 5 divide into 25? We know that $5 \times 5 = 25$, so the result is 5 (written in the quotient) and since 25 – 25 = 0, the remainder is 0. This very quick division process allows us to see $19875 \div 5 = 3975$.

Step 1	***Step 2***	***Step 3***	***Step 4***	***Step 5***
$\begin{array}{r} 0 \\ 5\overline{)1\ 9\ 8\ 7\ 5} \end{array}$	$\begin{array}{r} 0\ 3 \\ 5\overline{)1\ 9_4 8\ 7\ 5} \end{array}$	$\begin{array}{r} 0\ 3\ 9 \\ 5\overline{)1\ 9_4 8\ {}_3 7\ 5} \end{array}$	$\begin{array}{r} 0\ 3\ 9\ 7 \\ 5\overline{)1\ 9_4 8\ {}_3 7\ {}_2 5} \end{array}$	$\begin{array}{r} 0\ 3\ 9\ 7\ 5 \\ 5\overline{)1\ 9_4 8\ {}_3 7\ {}_2 5} \end{array}$

EXPLORE! Try a few on your own using the Short Division algorithm.

A) $296 \div 4$

C) $9{,}143 \div 5$

B) $7{,}843 \div 6$

D) $31{,}467 \div 3$

EXPLORE! Which algorithm would you use to compute these quotients?

	Quotient	Scaffolding	Traditional Algorithm	Short Division
A) **	$7{,}843 \div 11$			
B)	$7{,}843 \div 13$			
C)	$43{,}843 \div 5$			
D)	$43{,}843 \div 12$			

EXPLORE! Now compute the actual quotients and remainders using the technique selected above.

A) ** $7{,}843 \div 11$

C) $43{,}843 \div 5$

B) $7{,}843 \div 13$

D) $43{,}843 \div 12$

While these are great ways to do division by hand, the calculator can also be helpful. However, sometimes the calculator gives a lot of information and we need to be able to interpret what the calculator is telling us.

Example: Find the quotient and remainder of $195 \div 8$ using your calculator.

Type $195 \div 8$ into your calculator and get $195 \div 8$ = 24.375. The "24" is the quotient (how many whole times 8 divides in evenly). So next we will see how to find the remainder. Press clear first. Then, find out what 24 × 8 is (because 24 groups of 8 will take up most of the 195). 24 × 8 = 192.

The remainder will be 195 – 192 = 3. Therefore, $195 \div 8$ = 24 R 3, or we could write this as an equation too: 195 = 24 × 8 + 3.

EXPLORE! Here's a few more to try with the calculator – find the quotient and remainder. When finished write as an equation like we did on the previous page: 195 = 24 × 8 + 3.

A) ** 🖩 $23{,}458 \div 13 =$

B) 🖩 $1{,}945 \div 7 =$

C) 🖩 $849{,}304 \div 185 =$

D) 🖩 $849{,}304 \div 19 =$

5.3: Division of Integers

Thankfully, division of integers is very similar to division of whole numbers. Since multiplication is the operation we used to define division, the same rules for signs in multiplication would apply to division.

Example: $(-18) \div 6 =$

In order to think of the quotient here, we should remember that whatever the result is, we could check it with multiplication. What number do we multiply by 6 to end with –18? Since –3 × 6 = – 18, and no other numbers work, we can say that $(-18) \div 6 = -3$.

As shown with multiplication,

- neg × neg = pos
- pos × neg = neg
- neg × pos = neg
- pos × pos = pos

With the connection to division, let's make the same connection with division. Take a moment and fill out the appropriate sign:

- neg ÷ neg = _____
- neg ÷ pos = _____
- pos ÷ neg = _____
- pos ÷ pos = _____

EXPLORE! Now try a few division problems with integers:

A) $(-45) \div (-9) =$

B) $72 \div (-9) =$

C) $(-45) \div 5 =$

D) $32 \div 8 =$

E) ** 🖩 $(-1{,}998) \div (-37) =$

F) 🖩 $198{,}084 \div (-68) =$

Concept Questions

Label each of the following expressions as P (always positive), S (sometimes positive and sometimes negative), or N (always negative). If the answer is P or N, explain why. But if the answer is S, give one example that shows the result could be positive and one example that shows the result could be negative.

Expression	Sign (circle one)	Examples or Explanation
pos ÷ pos	P S N	
pos ÷ neg	P S N	
neg ÷ neg	P S N	
neg ÷ pos	P S N	
pos × pos	P S N	
pos × neg	P S N	
neg × neg	P S N	
neg × pos	P S N	
Expression	**Sign (circle one)**	**Examples or Explanation**
pos + neg	P S N	
neg + pos	P S N	
neg + neg	P S N	
neg – pos	P S N	
neg – neg	P S N	
pos – pos	P S N	

Explain why addition and subtraction are more challenging when integers are involved.

5.4: Division of Fractions

Multiplication of fractions was fairly convenient: we multiply "straight across." Wouldn't it be really nice if division of fractions was just as quick? Well, perhaps it is.

Example 1: $\left(\frac{16}{25}\right) \div \left(\frac{2}{5}\right) = \underline{\qquad}$

Find the quotient, and check your result using multiplication: $\left(\frac{2}{5}\right) \times \underline{\qquad} = \left(\frac{16}{25}\right)$. Did it work out?

Of course, it could just be a fluke… so let's try a couple more. Be sure to check your results with multiplication.

A) $\left(\frac{36}{25}\right) \div \left(\frac{4}{5}\right) = \underline{\qquad}$

B) $\left(\frac{45}{28}\right) \div \left(\frac{5}{7}\right) = \underline{\qquad}$

This is pretty interesting, as it seems that division of fractions could be simply done by dividing straight across, just like multiplication. Here's a couple more that might give us some helpful techniques later.

C) $\left(\frac{36}{17}\right) \div \left(\frac{4}{17}\right) = \underline{\qquad}$

D) $\left(\frac{45}{7}\right) \div \left(\frac{9}{7}\right) = \underline{\qquad}$

This leads to a very interesting result for when two fractions have the same denominator: $\left(\frac{a}{b}\right) \div \left(\frac{c}{b}\right) = \frac{a}{c}$. This may seem strange, but it is very cool and fast. We can now divide quickly.

EXPLORE! Divide the following using our new rule: $\left(\frac{a}{b}\right) \div \left(\frac{c}{b}\right) = \frac{a}{c}$. Simplify if possible.

A) ** $\left(\frac{19}{8}\right) \div \left(\frac{7}{8}\right) = \underline{\qquad}$

C) $\left(\frac{60}{49}\right) \div \left(\frac{20}{49}\right) = \underline{\qquad}$

B) $\left(\frac{15}{237}\right) \div \left(\frac{7}{237}\right) = \underline{\qquad}$

D) $\left(\frac{461}{725}\right) \div \left(\frac{900}{725}\right) = \underline{\qquad}$

Let's try a new problem that might be what some of you are thinking about… what happens when the division doesn't go the way we want it to? What happens when the denominators are not the same?

<u>Example:</u> $\left(\frac{3}{5}\right) \div \left(\frac{4}{7}\right) = \underline{\qquad}$

This problem is a bit more challenging since the 4 doesn't divide into the 3 evenly, and the 7 doesn't divide evenly into the 5… which presents a problem of sorts. If only we had a tool that would allow us to rewrite fractions, a way to rewrite them so that the division would work quickly! We do – and it's called FLOF! Create a common denominator (like we did with addition/subtraction) and then see if we can do the rest of the division quickly.

$$\left(\frac{3}{5}\right) \div \left(\frac{4}{7}\right) = \left(\frac{3 \times 7}{5 \times 7}\right) \div \left(\frac{4 \times 5}{7 \times 5}\right) = \left(\frac{21}{35}\right) \div \left(\frac{20}{35}\right) = \underline{\qquad}$$

<u>EXPLORE!</u> When we take the time to get a common denominator, the division works very quickly. Of course, it could just be a fluke… so let's try a couple more. Be sure to check your results with multiplication and simplify the quotients when possible.

A) ** $\left(\frac{28}{9}\right) \div \left(\frac{5}{3}\right) = \underline{\qquad}$ (Do this one with a large common denominator)

B) $\left(\frac{28}{9}\right) \div \left(\frac{5}{3}\right) = \underline{\qquad}$ Do this one with the smallest common denominator – does the answer match part (A)?

C) $\left(\frac{2}{3}\right) \div \left(\frac{4}{5}\right) = \underline{\qquad}$

Now we will come up with a rule about how to do this last type of division, and show it works using multiplication (we'll get the common denominator first):

$$\left(\frac{a}{b}\right) \div \left(\frac{c}{d}\right) = \left(\frac{ad}{bd}\right) \div \left(\frac{bc}{bd}\right) = \frac{ad}{bc}$$

We will call this the <u>LCD and Divide Straight Across Method.</u> Some just call it "awesome!"

This is where things get interesting. With multiplication of fractions, we know it is done straight across. And since we "*Simplify before we multiply*," there are some problems that come up and we could solve them now. Fill in the blank to find the value that makes the equation true:
$\left(\frac{2}{9}\right) \times \left(\overline{\qquad}\right) = 1$.

Try a few more: $\left(\frac{3}{7}\right) \times \left(\overline{\qquad}\right) = 1$ $\qquad$ $\left(-\frac{15}{11}\right) \times \left(\overline{\qquad}\right) = 1$

When we start with a number "*a*" and multiply it by another number to get 1, there is a special term for that.

Based on these problems, it seems that for almost every number, there is another number that we could multiply to make 1. In general, this rule is called the <u>**Multiplicative Inverse Property**</u>: $a \cdot b = b \cdot a = 1$, and *a* and *b* are called <u>**multiplicative inverses (or reciprocals)**</u>.

<u>**EXPLORE!**</u> Find the multiplicative inverse of the following numbers, <u>*if possible*</u>:

A) $\frac{5}{17}$

D) $\frac{1}{11}$

B) -14

E) 0

C) $-\frac{9}{14}$

F) 1

Example: Describe what happened with the last part above – were you able to find the multiplicative inverse of 0? Try to explain why or why not.

What number can we multiply by 0 and get to 1? That's the same type of question we had in the first few sections when the issue was 1 ÷ 0? If you recall, this was a situation where the answer didn't exist! There is no number that we can multiply by 0 to get to 1, which is why 1 ÷ 0 is undefined. That means there is no multiplicative inverse for 0.

NOTE: It seems that this is our first property that doesn't always work – every number has a multiplicative inverse except for one number… zero. Typically, we have properties in mathematics that always work and if they don't always work, we don't use them. But this Multiplicative Inverse Property is an exception. Because the only exclusion is 0, we allow it and move forward. ☺

Now there's a nice coincidence that links this new property to the multiplication and division we've already done. From the divide straight across formula, we know that

$\left(\frac{a}{b}\right)\div\left(\frac{c}{d}\right)=\left(\frac{ad}{bd}\right)\div\left(\frac{bc}{bd}\right)=\frac{ad}{bc}$. But look at the final quotient, $\frac{ad}{bc}$ … we could rewrite that as multiplication: $\frac{ad}{bc}=\frac{a\times d}{b\times c}=\left(\frac{a}{b}\right)\times\left(\frac{d}{c}\right)$.

This is fairly helpful as it shows that we could rewrite division as multiplication directly.
$\left(\frac{a}{b}\right)\div\left(\frac{c}{d}\right)=\left(\frac{a}{b}\right)\times\left(\frac{d}{c}\right)$.

In general, we could rewrite division as multiplication by the multiplicative inverse (reciprocal) or we could divide straight across. At this point you might notice a very strong correspondence between this division property and the subtraction property from the previous unit. How can we rewrite $a-b$?

The following equations demonstrate the two similar properties. Since the field properties are addition and multiplication, and since subtraction and division are defined in terms of their inverse operations, the ability to rewrite subtraction and division in terms of addition and multiplication will be very helpful.

Subtraction/Addition Connection

- $a-b=a+(-b)$
- We can rewrite subtraction as adding the additive inverse (opposite).
- 0 is the key number and splits the signs
  - Greater than 0 is positive
  - Less than 0 is negative
- Adding a positive makes things bigger
- Adding a negative makes things smaller

Division/Multiplication Connection

- $a\div b=a\times\left(\frac{1}{b}\right)$
- We can rewrite division as multiplying by the multiplicative inverse (reciprocal).
- 1 is the key number for these operations
- Multiplying by a size greater than 1 makes things bigger
- Multiplying by a size smaller than 1 makes things smaller.

EXPLORE! Practice dividing fractions using one of the techniques shown here: (1) dividing straight across or (2) rewriting as multiplying by the reciprocal. Be sure to simplify the quotient, if possible.

A) ** $\left(\frac{5}{14}\right) \div \left(\frac{1}{7}\right)$

C) $\left(\frac{26}{11}\right) \div \left(-\frac{10}{33}\right)$

B) $\left(-\frac{2}{3}\right) \div \left(-\frac{5}{9}\right)$

D) $\left(-\frac{187}{125}\right) \div \left(-\frac{33}{45}\right)$

The flexibility of FLOF is that it can be used outside of fractions to rewrite quotients and make the division simpler.

Example: $200 \div 12$ can be written as a fraction like $\frac{200}{12}$ and then simplified. $\frac{200}{12} = \frac{4 \cdot 50}{4 \cdot 3} = \frac{50}{3}$.

What this means is that $200 \div 12 = 50 \div 3$. For most people, dividing by 3 is easier than dividing by 12. Because of the way that FLOF works, there are many ways to rewrite $200 \div 12$ so that the resulting division would produce similar results.

- $200 \div 12$
- $400 \div 24$
- $100 \div 6$
- $50 \div 3$

EXPLORE! Rewrite the division problem to make it easier using the techniques shown here. Show at least two different divisions for each.

A) ** $8{,}200 \div 400 =$

B) $(-200) \div (-45) =$

C) $3{,}200 \div 20 =$

Create a rule to tell others how they could modify a division problem that will be easier, but have the same quotient.

5.5: Division of Decimals

When thinking of division with decimals, a problem like 8,200 ÷ 400 comes to mind. Thankfully, the FLOF idea from the previous section allows us to rewrite this division with different values but the same quotient.

FLOF was originally written like this: $\frac{a}{b} = \frac{a \times n}{b \times n}$. But we can rewrite this as $a \div b = (an) \div (bn)$. This is why 8,200 ÷ 400 = 820 ÷ 40 = 82 ÷ 4 = 8.2 ÷ 0.4 = 0.82 ÷ 0.04 … and so on.

Example: 2.48 ÷ 0.05 can be rewritten by multiplying both numbers by 10 repeatedly until we find a division problem that is preferred. Usually having a small whole number divisor is best.

2.48 ÷ 0.05 = 24.8 ÷ 0.5 = 248 ÷ 5

This last one is much easier. We will use the space below to perform the long division, and put the decimal point in the correct place.

$0.05\overline{)2.48}$ is the starting point, but we'll rewrite it as $5\overline{)248}$. The division is shown to the right.

$$\begin{array}{r} 49 \\ 5\overline{)\;248} \\ \underline{-20} \\ 48 \\ \underline{-45} \\ 3 \end{array}$$

We can also see that $2.48 \div 0.05 = \frac{248}{100} \div \frac{5}{100} = \frac{248}{5}$. (using divide straight across)

While we could stop here and state that the result is 49 with a remainder of 3. But if we put in the decimal point, the division could continue because 248 = 248.0 = 248.00 = 248.000. We could write in as many 0's after the decimal point as we wand, just making sure to not change the value.

$$\begin{array}{r} 49.6 \\ 5\overline{)\;248.0} \\ \underline{-20} \\ 48 \\ \underline{-45} \\ 30 \\ \underline{-30} \\ 0 \end{array}$$

In decimal form, 2.48 ÷ 0.05 = 49.6. We could confirm this with multiplication by hand, or with a calculator.

Typically, it is easiest to perform division when the divisor is a whole number, so we can multiply or divide by 10 until that happens. If you can go further and rewrite with a better divisor, you should!

EXPLORE! Practice rewriting the following divisions with nicer (whole number) divisors. At this point, we won't perform the actual calculation.

A) $8.23 \div 0.2 =$

B) $0.39 \div 5.1 =$

C) $0.0003 \div 0.007 =$

D) $17 \div 0.005 =$

Sometimes, the quotient seems to continue on forever. When the pattern repeats, like $2 \div 3$, we write the repeating portion with a bar over it. This portion is called the **repetend**. $2 \div 3 = 0.\overline{6}$.

Since fractions can be written as division, this gives a process for converting fractions into decimals.

Interactive Example: Write as a decimal: $\frac{5}{6}$.

EXPLORE! Perform the division by hand unless indicated with the calculator symbol. If the value is a fraction, convert the fraction to division and then write as a decimal.

A) $17 \div 0.005 =$

B) $\frac{5}{8} =$

C) $9.3 \div 0.22 =$

D) 🖩 $0.009 \div 0.37 =$

E) 🖩 $\frac{5}{6} =$

5.6: Division with Exponents

In a previous section, we saw that $x^A \cdot x^B = x^{A+B}$ and also that with FLOF, $\frac{a \times n}{b \times n} = \frac{a}{b}$. How could we combine these ideas with $\frac{x^5}{x^3}$?

Example: Simplify $\frac{x^5}{x^3}$.

It may help here to write this out without exponents: $\frac{x^5}{x^3} = \frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x \cdot x} = \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x}} = \frac{x \cdot x}{1} = x^2$.
Since there are common factors in both the numerator and denominator, we could simplify those common factors and end up with the result.

EXPLORE! Practice using your rule – write the result using exponents.

A) ** $\frac{4^9}{4^7} =$

C) $\frac{18^{55}}{18^{55}} =$

B) $\frac{427^{32}}{427^{15}} =$

D) $\frac{2^{53}}{2^{54}} =$

Create a rule for $\frac{x^A}{x^B} =$

The last two examples bring up some interesting ideas: $\frac{18^{55}}{18^{55}} = 18^0$ from our rule. But what does a number raised to the 0 power mean? Let's look deeper.

Example: $\frac{7^2}{7^2} = 7^0$ from our rule, but what if we expanded the powers out: $\frac{7^2}{7^2} = \frac{49}{49}$. Simplifying this, we can see that $\frac{7^2}{7^2} = \frac{49}{49} = 1$, but our rule shows $\frac{7^2}{7^2} = 7^0$. This means that the two results must be the same, which means $7^0 = 1$. Use your calculator to confirm this fact.

EXPLORE! Find the value of the following expressions.

A) ** $2.483^0 =$

C) $(-5)^0 =$

B) $-13^0 =$

D) $23^0 =$

The base is important to finding the value of an exponential expression.

- What is the base of $(-5)^0$?
- What is the base of -7^0?

We could read the -7^0 to be the same as $-(7^0)$, and is best thought of as the opposite of 7^0. One way to think of x^0 is as $x^0 = \frac{x}{x}$, or a number divided by itself. For this reason, $x^0 = 1$ *nearly* all the time.

Interactive Example: How could we write 0^0? Determine whether this has a value or not.

With this in mind, we could say $x^0 = 1$ unless: ________________________.

For Love of the Math: *While we are using "undefined" for situations like this, many mathematicians prefer "indeterminate" for situations where the condition of uniqueness fails.*

Now we'll discover another great usefulness with exponents. From a previous problem, we saw $\frac{2^{53}}{2^{54}} = 2^{-1}$. But what does a negative exponent represent?

Example: What does 2^{-1} represent?

To determine the value of 2^{-1}, let's multiply it by something and see what happens. For this example, we'll multiply by 2.

Determine the value: $(2^{-1})(2) =$

But from the multiplicative inverse, we know that there is already something to multiply by 2 and end with 1… the reciprocal.

Determine the value: $\left(\frac{1}{2}\right)(2) =$

Since both of these equations are equal to 1, we can say that $2^{-1} = \frac{1}{2}$.

Create a rule for $x^{-1} =$

Will the rule hold true for every value of *x*, or is there a value that will cause this to fail?

EXPLORE! Rewrite using positive exponents.

A) ** x^{-2}

B) x^{-5}

In one manner of thinking, the negative exponent is the opposite of a positive exponent. Since a positive exponent is multiplying by the base repeatedly, a negative exponent would be dividing by the base repeatedly. And dividing by a number is the same as multiplying by the reciprocal – so others refer to a negative exponent as being the reciprocal of the base. Both ideas will get you the same result:

$$\left(\frac{2}{3}\right)^{-2} = \frac{1}{\left(\frac{2}{3}\right)^{2}} = \frac{1}{\left(\frac{4}{9}\right)} = 1 \div \left(\frac{4}{9}\right) = 1 \times \left(\frac{9}{4}\right) = \frac{9}{4} \quad \text{or} \quad \left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^{2} = \frac{9}{4}$$

EXPLORE! Simplify and rewrite the following with positive exponents.

A) ** $x^{15} \cdot x^{-21}$

D) $(-5)^{-2}$

B) $\left(-\frac{3}{7}\right)^{-2}$

E) 0^{-7}

C) -5^{-2}

F) $\left(\frac{2x}{y^2}\right)^{-3}$

<u>Scientific Application:</u> Negative and positive exponents allow us to rewrite large and small numbers in a way that makes them easier to handle. There are many ways to rewrite a number like 14,000.

	Written as standard multiplication	**Written as multiplication with exponents on 10**
A)	$14{,}000 \times 1$	$14{,}000 \times 10^0$
B)	1400×10	1400×10^1
C)	140×100	140×10^2
D)	$14 \times 1{,}000$	14×10^3
E)	$1.4 \times 10{,}000$	1.4×10^4
F)	$0.14 \times 100{,}000$	0.14×10^5

Since every expression in the last table is equal to 14,000, scientists had to choose the one that was easiest to work with. Standard multiplication becomes tedious when there are a large number of 0's in a number, so the exponent on 10 method was chosen; scientists write numbers so they look like $a \times 10^n$.

And the choice for the value of *a*? Scientists picked the version of the number so $1 \le |a| < 10$. From our previous table, only one number meets that condition: 1.4×10^4.

Writing a number in the form $a \times 10^n$ where $1 \le |a| < 10$ and *n* is an integer, is called <u>**scientific notation**</u>. Remember that many other formats will have the same value, but only one is scientific notation.

Let's do one more table for the number 0.00078. Remember that $10^{-2} = \frac{1}{10^2} = \frac{1}{100}$.

	Written as standard multiplication	**Written as multiplication with exponents on 10**
A)	0.00078×1	0.00078×10^0
B)	$0.0078 \times \frac{1}{10}$	0.0078×10^{-1}
C)	$0.078 \times \frac{1}{100}$	0.078×10^{-2}
D)	$0.78 \times \frac{1}{1{,}000}$	0.78×10^{-3}
E)	$7.8 \times \frac{1}{10{,}000}$	7.8×10^{-4}
F)	$78 \times \frac{1}{100{,}000}$	78×10^{-5}

Which of the formats in the table above is in scientific notation?

With a few examples under our belt, let's categorize the following scientific notation numbers as either big or small depending on whether the size (absolute value) is bigger than 1 or smaller than 1.

EXPLORE! Where are these numbers on a number line?

	Number	Sign	Size
A) **	1.9×10^{24}	Left of zero Right of zero	Far from zero Close to zero
B)	-3.8×10^{79}	Left of zero Right of zero	Far from zero Close to zero
C)	2.5×10^{-54}	Left of zero Right of zero	Far from zero Close to zero
D)	-1.9×10^{-304}	Left of zero Right of zero	Far from zero Close to zero

EXPLORE! Fill in the table by writing the value in either standard notation or scientific notation.

	Standard Notation	Scientific Notation
A)	17,000	
B)	0.00000048	
C)		7.8×10^{-4}
D)		4.35×10^{5}

There are some guiding principles here that relate to compensation and keeping the same value; these ideas are seen in the previous tables:

- when we increase the size of the initial factor, then the power of 10 must (increase / decrease).
- when we decrease the size of the initial factor, then the power of 10 must (increase / decrease).

EXPLORE! Determine which numbers are written in scientific notation. If it is not written in scientific notation properly, rewrite the number properly in scientific notation – but keep the same value!

A) ** 147×10^{43}

B) -1.8×10^{-84}

C) $1{,}700\times10^{-20}$

D) -0.000035×10^{49}

E) -0.0085×10^{-49}

F) 3.5×10^{49}

5.7: Number Sense and Division

When we divide numbers, many people think that the size goes down. And in one manner of thinking, it does: $27 \div 9$ will be smaller than 27.

What about $8 \div 0.5$? When we divide that out, we find $8 \div 0.5 = 16$ which is larger than 8. The size of the quotient depends on the divisor, but the key idea to understand relates to 1 again. The multiplicative identity property shows $a \times 1 = a$, which means $a \div 1 = a$. Dividing any number by 1 doesn't change the value because the quotient and dividend are the same.

Think back to what we've learned about the sign of the quotient?

A) pos ÷ pos =

B) pos ÷ neg =

C) neg ÷ pos =

D) neg ÷ neg =

Once the sign is determined, we can focus on the sign of the quotient by dividing the absolute values (or sizes). But what happens to the size (absolute value) of the product?

Interactive Examples: circle whether the product is greater or less than the values listed.

	If the problem is…	**Then the quotient will be …**	
A) **	$8 \div 6$	Greater Less than 1	Greater Less than 8
B)	$8 \div 0.7$	Greater Less than 1	Greater Less than 8
C)	$14 \div 0.25$	Greater Less than 1	Greater Less than 14
D)	$14 \div 19.25$	Greater Less than 1	Greater Less than 14

EXPLORE! To summarize the results, we can make a list of what we saw:

	If the size of B is…	**Then the size of A ÷ B will be …**			
A) **	0	Larger than A	Smaller than A	Equal to A	Undefined
B)	Between 0 and 1	Larger than A	Smaller than A	Equal to A	Undefined
C)	1	Larger than A	Smaller than A	Equal to A	Undefined
D)	Greater than 1	Larger than A	Smaller than A	Equal to A	Undefined
E)	Greater than A	Larger than 1	Smaller than 1	Equal to 1	Undefined
F)	Less than A	Larger than 1	Smaller than 1	Equal to 1	Undefined
G)	Equal to A	Larger than 1	Smaller than 1	Equal to 1	Undefined

EXPLORE! Determine the size and sign of the quotient without performing actual division.

	Quotient	Sign	Size is…
A) **	$\frac{2}{3} \div (-5)$	Pos Neg Zero	Greater than $\frac{2}{3}$ Less than $\frac{2}{3}$
B)	$\left(-\frac{2}{3}\right) \div \left(-\frac{7}{9}\right)$	Pos Neg Zero	Greater than $\frac{2}{3}$ Less than $\frac{2}{3}$
C)	$(-11) \div \left(-\frac{7}{9}\right)$	Pos Neg Zero	Greater than 11 Less than 11
D)	$(-11) \div (1.05)$	Pos Neg Zero	Greater than 11 Less than 11
E)	$(6.92) \div (3.25)$	Pos Neg Zero	Greater than 6.92 Less than 6.92
F)	$(-0.92) \div \left(\frac{1}{2}\right)$	Pos Neg Zero	Greater than 0.92 Less than 0.92

EXPLORE! Determine the size of the quotient without performing actual multiplication.

	Quotient	Sign is…	Size is…
G) **	$11 \div (\text{Number between 0 and 1})$	Pos Neg	Greater than 11 Less than 11
H)	$11 \div (\text{Number greater than 1})$	Pos Neg	Greater than 11 Less than 11
I)	$11 \div (\text{Number less than } -1)$	Pos Neg	Greater than 11 Less than 11
J)	$11 \div (\text{Number between } -1 \text{ and } 0)$	Pos Neg	Greater than 11 Less than 11

These are really important parts to understand – division and multiplication are based on relationship to the number 1.

- Multiply by size bigger than 1, gets larger.
- Multiply by size smaller than 1, gets smaller.

For division, the reverse is true (as multiplication is the inverse of division).

- Divide by size bigger than 1, gets smaller.
- Divide by size smaller than 1, gets larger.

We will apply these to number lines in the next section!

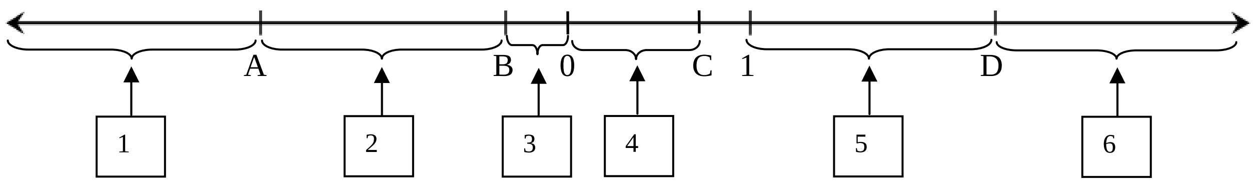
5.8: Division with Number Lines

Since division can be represented with fractions, we could use number lines to represent the resulting values.

Example: Find the region that best approximates 1 ÷ C.

Our solution will relate to the size and sign of the numbers.

- To find the value, we can first think of the sign. 1 is positive and C is positive, so the quotient will be positive.
- Next, we need to think about the size. Since C is smaller than 1, 1 ÷ C must get bigger.
    - One group of C is drawn below and won't get us all the way to 1. So we know that the result must be more than 1 group of C.
- The solution would be in region 5… although, if you wrote region 6, you would get some credit.



EXPLORE! Practice more with the number lines and be cautious with the sign!

A) ** Find the region that best approximates 1 ÷ D.

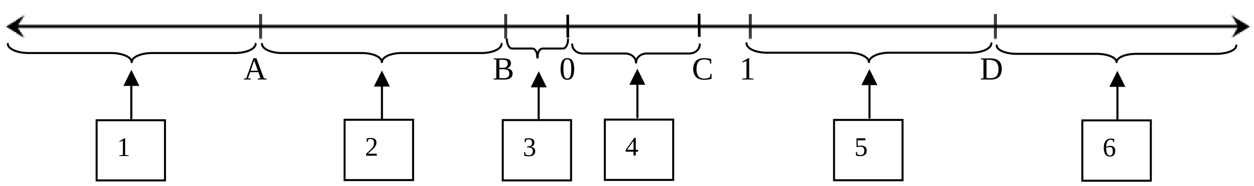
B) ** Find the region that best approximates C ÷ B.

C) Find the region that best approximates 1 ÷ B.

D) Find the region that best approximates C ÷ D.

E) Find the region that best approximates B ÷ A.

F) Find the region that best approximates D ÷ C.



5.9: Mixed Numbers and Improper Fractions

Now that we have skills working with fractions and the operations of addition, subtraction, multiplication, and division, we can take on the larger idea of how we write fractions.

To this point, we haven't really worried about the size of the numerator. $\frac{22}{13}$, $-\frac{13}{13}$, and $\frac{5}{13}$ are all types of fractions. In order to specify the sizes when discussing a general fraction like $\frac{a}{b}$, we use the following terms:

- $\frac{a}{b}$ is called a **proper** fraction if $|a| < |b|$.
- $\frac{a}{b}$ is called **improper** fraction if $|a| \geq |b|$.

EXPLORE! Identify the following fractions as proper or improper.

	Fraction	Label
A) **	$\frac{7}{5}$	Proper Improper
C)	$\frac{-11}{52}$	Proper Improper
E)	$\frac{7}{-52}$	Proper Improper

	Fraction	Label
B) **	$\frac{-11}{5}$	Proper Improper
D)	$\frac{-11}{-52}$	Proper Improper
F)	$\frac{8}{23}$	Proper Improper

Sometimes people will refer to an improper fraction as a fraction where the "top is bigger than the bottom." Based on the table above, does this slang actually hold true? Explain your conclusion.

As we have seen, $\frac{3}{3}=1$ because $3 \div 3 = 1$. When we had larger division problems like $22 \div 3$, we had written this quotient as $7 \text{ R } 1$. But $22 \div 3 = \frac{22}{3}$, and when we consider 22 objects that are all thirds, each group of 3 makes up one whole unit. $\frac{22}{3}$ has 7 whole units, but this doesn't use up all of the 22 pieces, there is still one left. Since we were working with thirds, this one extra piece is one-third, and we could really write $\frac{22}{3} = 7 + \frac{1}{3}$. Because this number has both integer and fraction parts, we call it a **mixed number**. Often, this is written without the addition symbol: $\frac{22}{3} = 7\frac{1}{3}$.

EXPLORE! Convert the fractions from improper to mixed number, or vice-versa.

	Fraction Notation	Mixed Number Notation
A) **	$-\frac{67}{9}$	
C)		$2\frac{13}{17}$

	Fraction Notation	Mixed Number Notation
B)	$\frac{60}{5}$	
D)		$-3\frac{2}{7}$

Go back to the table and try to convert using your calculator. There are two ways – one is using the fraction buttons and the other is using division and finding the remainder.

Caution: $-3\frac{2}{7}$ is one that tends to create problems. Many students multiply $(-3)\times 7 = -21$, then add 2 which results in – 21 + 2 = – 19. As teachers, we often see $-3\frac{2}{7} = -\frac{19}{7}$ … which is *not correct*. ☹ However, we have to show the connection: $-3\frac{2}{7}$ is the opposite of $3\frac{2}{7}$, so if we find $3\frac{2}{7}$, it may be quicker to just take the opposite of that. $3\frac{2}{7}$ is found with $3\times 7 = 21$, and 21 + 2 = 23, so $3\frac{2}{7} = \frac{23}{7}$. This shows $-3\frac{2}{7} = -\frac{23}{7}$ which is the correct value. Be careful working with negative mixed numbers!

Review from previous sections: What operations with fractions do we need to have common denominators… and why?

	Operation	Example	Needs common denominator?
A) **	Addition	$\frac{2}{7}+\frac{3}{5}$	YES NO
B)	Subtraction	$\frac{2}{7}-\frac{3}{5}$	YES NO
C)	Multiplication	$\frac{2}{7}\times\frac{3}{5}$	YES NO
D)	Division	$\frac{2}{7}\div\frac{3}{5}$	YES NO

When doing addition or subtraction with mixed numbers, there are some options for how to do it. We could convert to improper or leave as mixed numbers. Each option is shown below:

Improper Fraction Method	Mixed Number Method
$2{,}301\frac{2}{5}+7{,}001\frac{3}{4}=$	$2{,}301\frac{2}{5}+7{,}001\frac{3}{4}=$
$\frac{11{,}507}{5}+\frac{28{,}007}{4}=$	$2{,}301+7{,}001+\frac{2\cdot 4}{5\cdot 4}+\frac{3\cdot 5}{4\cdot 5}=$
$\frac{11{,}507\cdot 4}{5\cdot 4}+\frac{28{,}007\cdot 5}{4\cdot 5}=$	$9{,}302+\frac{8}{20}+\frac{15}{20}=$
$\frac{46{,}028}{20}+\frac{140{,}035}{20}=$	$9{,}302+\frac{23}{20}=$
$\frac{186{,}063}{20}=$	$9{,}302+1\frac{3}{20}=$
$9{,}303\frac{3}{20}$	$9{,}303\frac{3}{20}$

Which of the methods shown above would be better for you to use with $2{,}301\frac{2}{5}+7{,}001\frac{3}{4}$?

<u>EXPLORE!</u> Perform the operations shown below – then check with the calculator.

A) ** $19\frac{3}{4}-42\frac{1}{2}$

C) $2\frac{3}{4}+19\frac{1}{2}$

B) $32\frac{1}{4}+19\frac{1}{2}$

D) $-32\frac{1}{4}+19\frac{5}{6}$

So addition and subtraction might be easiest done with the mixed fraction form; but what about multiplication? There are two ways to do these problems as well – keeping in mixed number, and converting to improper. Let's see an example of how each method would work.

Improper Fraction Method	Mixed Number Method
$\left(5\frac{1}{3}\right)\left(3\frac{2}{5}\right)=$ $\left(\frac{16}{3}\right)\left(\frac{17}{5}\right)=$ $\frac{272}{15}=$ $18\frac{2}{15}$	$\left(5\frac{1}{3}\right)\left(3\frac{2}{5}\right)=$ $\left(5+\frac{1}{3}\right)\left(3\frac{2}{5}\right)=$ $5\left(3\frac{2}{5}\right)+\frac{1}{3}\left(3\frac{2}{5}\right)=$ $5\left(3+\frac{2}{5}\right)+\frac{1}{3}\left(3+\frac{2}{5}\right)=$ $5\times3+5\times\frac{2}{5}+\frac{1}{3}\times3+\frac{1}{3}\times\frac{2}{5}=$ $15+2+1+\frac{2}{15}=$ $18+\frac{2}{15}=$ $18\frac{2}{15}$

Both methods will get you to the correct answer for $\left(5\frac{1}{3}\right)\left(3\frac{2}{5}\right)$, but which method would you prefer to use?

EXPLORE! Perform the operations shown below using either method.

A) ** $2\frac{3}{4}\times9\frac{3}{11}$

B) $7\frac{1}{2}\times4\frac{2}{3}$

C) $2\frac{3}{4}\div3\frac{3}{11}$

D) $10\frac{1}{2}\div4\frac{2}{3}$

This really is like a life lesson about choosing the appropriate path or perhaps the most efficient path. Each person gets to pick their own path, but that decision has consequences. Choosing the long path here will cost you time and sometimes lots of time!

Which option (mixed number or improper fraction) seems to be most efficient when adding or subtracting mixed numbers? Why?

Which option (mixed number or improper fraction) seems to be most efficient when multiplying or dividing mixed numbers? Why?

<u>Could</u> you find the sum of $2{,}000\frac{7}{11}+41{,}000\frac{7}{22}$ by converting to improper fractions? YES NO

<u>Would</u> you find the sum of $2{,}000\frac{7}{11}+41{,}000\frac{7}{22}$ by converting to improper fractions? Explain.

Estimating is a way to get close to the answer quickly. For estimating, we don't care about what the actual value is, but just about getting close to it.

	Estimating the value of …	**Is closest to…**
A) **	$36{,}497 \div 219$	1.8 18 180 1,800
B)	$7\frac{1}{2}\times 4\frac{2}{3}$	3 30 300 3,000
C)	$490 \div 1.1$	0.49 4.9 49 490 4,900
D)	$490 \div 0.12$	0.49 4.9 49 490 4,900
E)	$\sqrt{62}$	5 6 7 8 9
F)	$7\frac{1}{2}\div 14\frac{2}{3}$	5 0.5 0.05 0.005
G)	$396{,}497{,}855 \div 798$	50 500 5,000 50,000 500,000
H)	$57\frac{1}{2}+34\frac{2}{3}$	9 90 900 9,000

Estimating helps us to spot if we typed something into the calculator incorrectly, which might save us points on an exam!

5.10: Roots and Radicals

The concept of exponents is to have repeated multiplication: $7^2 = 7 \cdot 7 = 49$. With every other operation, we've seen an inverse operation that will 'undo' the first. Addition will undo subtraction, multiplication will undo division, but what will undo exponents? What operation will take 49 as an input and then return 7 as an output?

Back in Unit 2, we were approximating square roots like $\sqrt{36}$. Now that we are really focusing on the concepts, the **square root** of a quantity is a number that you would square to get back to that quantity. So what can we square to get back to 36?

There really are two square roots of 36, one is positive and one is negative. The positive root is called the **principal square root** and is non-negative. $\sqrt{36} = 6$. For the negative root, we would write it as $-\sqrt{36}$, which is – 6.

EXPLORE! Find the values given.

A) ** $-\sqrt{1}$

B) $\sqrt{400}$

C) $\sqrt{64}$

D) $-\sqrt{81}$

What if we tried to take a square root of $\sqrt{-16}$, where the radicand is negative?

For the numbers we have dealt with, what is the result for neg × neg? ______________

For the numbers we have dealt with, what is the result for pos × pos? ______________

Example: What type of number could be squared and become negative?

There is no number that we currently use (no real number) that could be squared and become negative. So for something like $\sqrt{-16}$, we would respond "not a real number."

Create a rule about when we can take a root of a negative number and get back a real number.

Here are some tables showing exponential form and the square root form.

Exponential Form	1^2	2^2	3^2	4^2	5^2	6^2	7^2	8^2	9^2	10^2	11^2
Value	1	4	9	16	25	36	49	64	81	100	121

Exponential Form	12^2	13^2	14^2	15^2	16^2	17^2	18^2	19^2	20^2
Value	144	169	196	225	256	289	324	361	400

Square Root Form	$\sqrt{1}$	$\sqrt{4}$	$\sqrt{9}$	$\sqrt{16}$	$\sqrt{25}$	$\sqrt{36}$	$\sqrt{49}$	$\sqrt{64}$	$\sqrt{81}$	$\sqrt{100}$
Value	1	2	3	4	5	6	7	8	9	10

Square Root Form	$\sqrt{121}$	$\sqrt{144}$	$\sqrt{169}$	$\sqrt{196}$	$\sqrt{225}$	$\sqrt{256}$	$\sqrt{289}$	$\sqrt{324}$	$\sqrt{361}$	$\sqrt{400}$
Value	11	12	13	14	15	16	17	18	19	20

EXPLORE! Use the tables above to find the values, if possible.

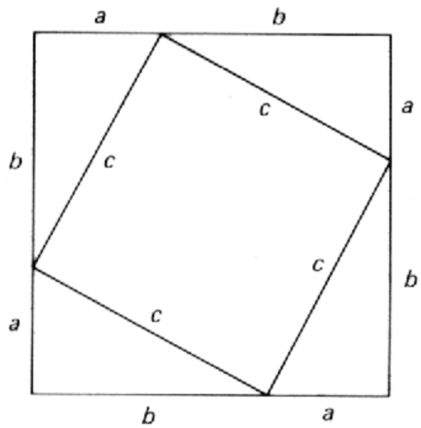
	A) **	B) **	C)	D)	E)	F)
Root Form	$-\sqrt{81}$	$\sqrt{-81}$	$\sqrt{-169}$	$-\sqrt{361}$	$\sqrt{225}$	$\sqrt{289}$
Value						

For Love of the Math: *Because there are two possible answers for equations like* $x^2 = 16$*, when solving these problems a square root is needed, and in higher classes, you'd see* $\sqrt{x^2} = |x|$*. Most of what we do in this class will deal with only the non-negative root.*

5.11: 2-Dimension Geometry and Division/Roots

We've seen the way that multiplication interacts with geometry, but exponents and roots come into play as well. Previously, we had dealt with perimeter and area, and area will allow us to come up with one of the most famous formulas in all of mathematics!

What we need for this is the area of a square $(A = b \cdot h)$ and the area of a triangle $\left(A = \frac{1}{2} b \cdot h\right)$. Look at the following shape:



There is a large outer square which is broken down into four **right triangles** (triangles with a 90˚ angle) and one smaller square. What we will do is compute the area in two different ways, and since the area is the same, regardless of how we break it down, we will be able to create a relationship between the two expressions.

Area in one way: Large Square

The large square has sides of $a+b$, so the area is $A = (a+b)(a+b)$. The distributive property allows us to write this area without parenthesis:

$$A = (a+b)(a+b) = (a+b)a + (a+b)b = a^2 + ba + ab + b^2 = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$$

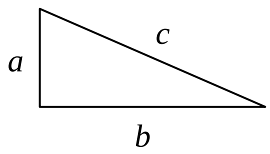
Area in another way: Four Triangles and Small Square

Each triangle has base of *b* and height of *a*, so each area is $A = \frac{1}{2} b \cdot a$. Since there are four of them, the sum of all four triangles area will be $A = 4\left(\frac{1}{2} b \cdot a\right) = 2ab$. Separately, the area of the smaller square is $A = c \cdot c = c^2$. The area of the entire shape is the sum of all of the pieces:

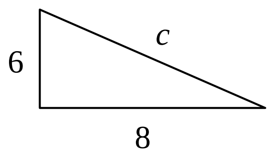
$$A = \underbrace{2ab}_{\text{Area of four triangles}} + \underbrace{c^2}_{\text{Area of small square}}$$

Conclusion: Since the area is the same both way, then we can say that $a^2 + 2ab + b^2 = 2ab + c^2$. Because they equal, we can remove equivalent expressions to get $a^2 + b^2 = c^2$.

If you have a right triangle, then the lengths of the **legs** (short sides) are related to the length of the **hypotenuse** (long side) by $a^2 + b^2 = c^2$.

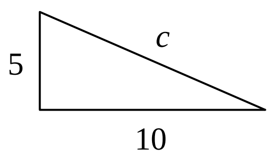


Example: In the following right triangle, find the missing piece.



Using the Pythagorean Theorem, we will put in the values of the legs as 6 and 8 (replacing *a* and *b*), and try to find the missing hypotenuse: *c*. $a^2 + b^2 = c^2 \Rightarrow 6^2 + 8^2 = c^2 \Rightarrow 36 + 64 = c^2 \Rightarrow 100 = c^2$. Once you have the equation $c^2 = 100$, we can take the square root and end up with $\sqrt{c^2} = \sqrt{100} \Rightarrow |c| = 10$. Since we are looking for a side length, and a length can never be negative, the only number that would work is 10.

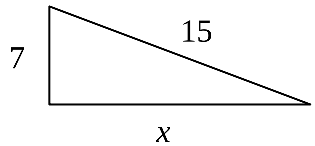
Interactive Example: In the following right triangle, find the missing piece. *Round to 3 decimal places if the square root is not a whole number.*



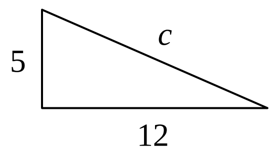
For Love of the Math: *The* ***Pythagorean Theorem*** *states: if a right triangle has legs of a and b, and hypotenuse c, then* $a^2 + b^2 = c^2$*. Most people remember just the last part, but the key is that this theorem only relates to right triangles and it will not be true on other triangles. There are many ways to prove this theorem, and Elisha Scott Loomis tried to put them all together in a book in 1927. He found about 370 different proofs, and there have been nearly 100 more since his book came out.*
http://files.eric.ed.gov/fulltext/ED037335.pdf

EXPLORE! In the following right triangle, find the missing piece. If the result is not a whole number, then use your calculator to find the length to 3 decimal places.

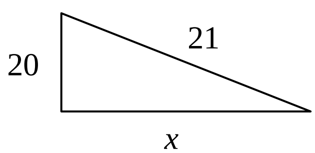
A) **



B)



C)



EXPLORE! Determine if the following sides will make a right triangle.

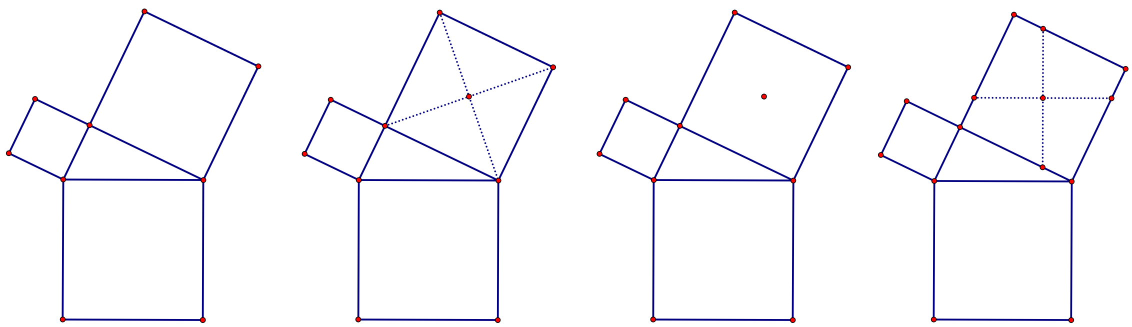
	Side Lengths	**Right Triangle?**
A) **	16, 63, and 65	Yes No
C)	11, 15, and 23	Yes No
E)	95, 168, and 183	Yes No

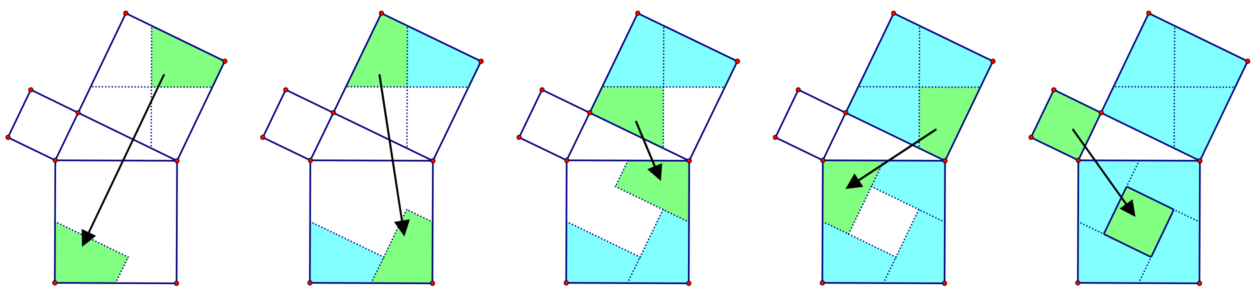
	Side Lengths	**Right Triangle?**
B) **	8, 15, and 17	Yes No
D)	20, 99, and 101	Yes No
F)	140, 171, and 221	Yes No

For Love of the Math:

The Pythagorean Theorem also links to the right triangle in a different way. The area of a square with a side length of x is $A = x \cdot x = x^2$. *Whenever we see an* x^2 *we can visualize a square with that side length. The Pythagorean Theorem has a lot of squares then, since* $a^2 + b^2 = c^2$. *If you think of a right triangle, then each side has a square associated with it.*

Here's a picture of a right triangle with squares drawn on each side (1). Then construct the center of the middle sized square by lightly drawing two lines, marking the point (2), and then erasing those lines (3). Then draw two lines, one parallel to the hypotenuse and one perpendicular (4). This forms four shapes with four sides called ***quadrilaterals****. One by one, take one quadrilateral and slide it into the larger square (5-8). Lastly, slide the area of the small square into the large square (9).*





This depiction shows that the area of the small square added to the area of the middle square (when cut up and reassembled) is equal to the area of the large square, which means $a^2 + b^2 = c^2$*!*

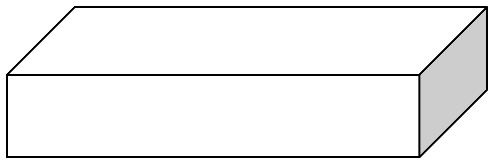
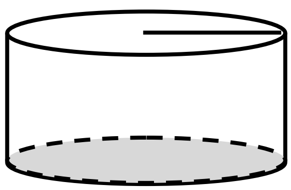
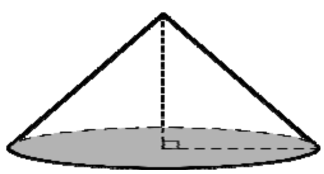
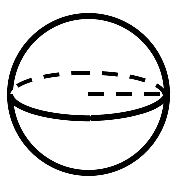
Henry Perigal wrote about this proof in 1891, the year he turned 90 years old! The idea is now known as Perigal's Dissection. https://www.youtube.com/watch?v=LtkAIQcACqY

You may ask yourself when this would be useful later in life, in life outside of a math class. Well, you may have a point. Sometimes we just look at really cool math stuff because it is really cool.

Extra Credit Challenge: *Build your own Perigal Dissection with cardboard or heavy paper.*

5.12: 3-Dimension Geometry and Division/Roots

3-dimensional shapes have formulas that are more complex, but they are still manageable. As you can see, these formulas use multiplication, exponents, and square roots to calculate different features of the shape. **Surface area** is the amount of area on all surfaces of the shape, while **volume** is the amount of 3-dimensional space that the shape occupies.

Picture of Shape	Formulas for the Shape
	Rectangular Solid $SA = 2l \cdot w + 2l \cdot h + 2w \cdot h$ $V = l \cdot w \cdot h$
	Circular Cylinder $SA = 2\pi \cdot r^2 + 2\pi \cdot r \cdot h$ $V = \pi \cdot r^2 \cdot h$
	Circular Cone $SA = \pi \cdot r^2 + \pi \cdot r \cdot \sqrt{r^2 + h^2}$ $V = \frac{1}{3}\pi \cdot r^2 \cdot h$
	Sphere $SA = 4\pi \cdot r^2$ $V = \frac{4}{3}\pi \cdot r^3$

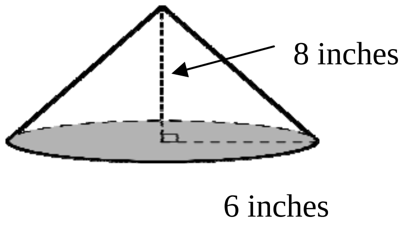
Example: So if we wanted to use a formula to find the volume of a sphere with radius of 15 inches, we would just take $V = \frac{4}{3}\pi \cdot r^3$ and put in the 15: $V = \frac{4}{3}\pi \cdot r^3 = \frac{4}{3}\pi \cdot (15)^3 = \frac{4}{3}\pi \cdot 3375 = 4500\pi$ in^3.

NOTE: The units are important with any of these calculations. Multiplying ft × ft × ft = ft^3.

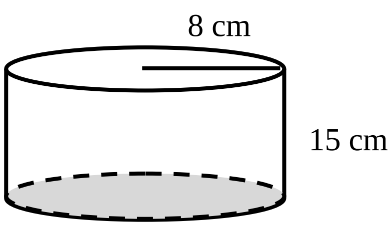
- So with volume, the units will most often be cubed (in^3, ft^3, yd^3).
- With area (or surface area), the units will most often be squared (in^2, ft^2, yd^2).
- With distance the units are most often just standard (inches, feet, yards, etc).

EXPLORE! Use the formulas on the previous page to find the surface area or volume of the 3-dimensional shapes.

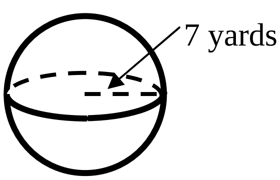
A) ** Find the surface area of this shape.



B) Find the volume of this shape.



C) Find the surface area of this shape.



5.13: Applications of Geometry, Division and Roots

Geometry Applications:

A) The San Diego Union-Tribune article on July 30 (about rain barrels), said that each 1,000 square feet of roof captures 625 gallons of water for every inch of rain that falls. Let's find out how much water we can capture from a roof that is 50 feet by 35 feet if one inch of rain falls.

Time/Date Applications:

A) It is currently Thursday. In 60 days, what day of the week will it be?

Money Applications:

A) If there is a salary of $52,000 per year for a job, about how much should you expect to make per month?

B) In a will, a parent leaves $27,200 for each of 3 kids to split. About how much would each child expect to receive?

C) If Angelica makes $3,000 per month, about how much is she earning per week?

D) If Angelica makes $877 per week, and works 40 hours per week, how much is she earning per hour?

5.14: Order of Operations

With so many operations at this point, we really need to get a handle on what they all have in common.

- Subtraction can be rewritten as adding the opposite, so subtraction and addition are really the same operation: Addition.
- Division can be rewritten as multiplying by the reciprocal, so multiplication and division are really the same operation: Multiplication – which is repeated addition.
    - *Both of these are field properties!*
- Exponents are repeated multiplication.
- Radicals or roots will undo exponents, which really is another way of thinking of exponents.

When we encounter certain mathematical expressions, the operations may seem to have different answers. Think about an expression like $5+2\times 8$.

$5+2\times 8$ might be thought of as $5+2\times 8=7\times 8=56$, but could also be thought of as $5+2\times 8=5+16=21$. Since there are multiple ways to do problems like this, one way to clear up the confusion is to use parenthesis. This way, the two forms are $(5+2)\times 8$ and $5+(2\times 8)$. While the parentheses option seems helpful, it becomes very tedious as we move to larger and more complex expressions like $(3\div(4-2))-(5+(2\times(7+9)))$.

This expression would be much simpler if we could identify an order to the operations that would help things look less complicated.

The lowest operation is performed last, with more complex operations being performed first. With this order of operations, we could write $(3\div(4-2))-(5+(2\times(7+9)))$ as $\frac{3}{4-2}-5+2(7+9)$. Notice there are far fewer parentheses, but there still could be some. Parentheses are a visual signal to indicate priority – parenthesis say "go here first". Parenthesis will take precedence over all other operations and absolute values are in the same level as parenthesis. This keeps our order or operations to be:

PEMA (Parenthesis, Exponents, Multiplication, Addition).

With this in mind, we've defined an order of operations:

1) Parenthesis – perform operations inside the parenthesis using the order of operations, and if there are no operations inside parenthesis, just a number, then we can move to the next operation.
2) Exponents or roots would be the first operation done.
3) Multiplication and division would be the next operation done – with all completed from left to right. We could always rewrite division as multiplication of the reciprocal at any point.
4) Addition and subtraction would be next, again with all completed from left to right. Likewise, we could also rewrite subtraction as adding the opposite at any stage.

EXPLORE: Do the parenthesis being included change the value of the expression?

	First Expression	Second Expression	Same or different
A) **	$8-5+1$	$(8-5)+1$	Same Different
B) **	$8-5+1$	$8-(5+1)$	Same Different
C)	$8-4$	$8(-4)$	Same Different
D)	$3(7)+4$	$37+4$	Same Different
E)	-4^2	$(-4)^2$	Same Different
F)	-2^3	$(-2)^3$	Same Different
G)	$7\left(\frac{2}{3}\right)$	$7\frac{2}{3}$	Same Different
H)	$8-5\times 7$	$8-(5\times 7)$	Same Different
I)	$8-5\times 7$	$(8-5)\times 7$	Same Different
J)	$8+12\div 3\times 2$	$8+12\div(3\times 2)$	Same Different
K)	$8+12\div 3\times 2$	$8+(12\div 3)\times 2$	Same Different
L)	0.35	$0\cdot 35$	Same Different
M)	$15+(n-1)-3$	$15+(n-1)(-3)$	Same Different

Example: Simplify $8+12\div 3\times 2$.

$8+12\div 3\times 2$ has three operations: addition, multiplication and division. Using PEMA, we can skip past the "P" and "E" and into multiplication. Division and multiplication are effectively the same, so we'll do those in order from left to right first. Once finished, we'll do the addition. Here's how it would look:

$8+12\div 3\times 2=$
$8+4\times 2=$
$8+8=$
16

EXPLORE: Simplify the following expressions using the order of operations.

A) ** $8-15\times11\div3+1$

D) $7-|-5|$

B) ** $18-15+(9-11)^2$

E) $8+12\div3\times2$

C) $18-(15+11)-9$

F) $3+8^2+12\div3\times2$

As in Unit 3, we could find values one step at a time using number boxes.

Interactive Example: Fill in the remaining boxes.

$$\boxed{1} \xrightarrow{\times 3} \boxed{} \xrightarrow{-4} \boxed{}$$

We could start in any place and work in either direction to fill in all empty boxes.

<u>EXPLORE!</u> Fill in the remaining boxes.

A) ** $\boxed{1} \xrightarrow{\times 3} \square \xrightarrow{-\,4} \square$

B) $\boxed{-5} \xrightarrow{-\,4} \square \xrightarrow{\times 3} \square$

C) $\square \xrightarrow{-\,4} \boxed{-5} \xrightarrow{\times 3} \square$

D) ** $\square \xrightarrow{+\,4} \square \xrightarrow{\div\,7} \boxed{-5}$

E) $\square \xrightarrow{+\,4} \boxed{2.1} \xrightarrow{\times 3} \square$

F) $\square \xrightarrow{\div\,(-3)} \boxed{7} \xrightarrow{+\,57} \square \xrightarrow{\sqrt{}} \square$

G) $\square \xrightarrow{\div\,5} \square \xrightarrow{-\,97} \square \xrightarrow{x^2} \boxed{361}$

Is there more than one solution for part (G)?

5.15: Division Summary

When dealing with division, the key is to think of it in terms of multiplication. Here are some wrap up questions to tie all the concepts together.

If you recall from multiplication, there were a number of properties that held true. But, do those same properties work with division?

- Is division commutative? Write a few equations that would be true if division was commutative, then explain your conclusion.

- Is division associative? Write a few equations that would be true if division was associative, then explain your conclusion.

- Is there a distributive property with division? Is either of these true:
    - $(a+b) \div c = a \div c + b \div c$
    - $c \div (a+b) = c \div a + c \div b$
    - Explain your conclusion.

- What is the Multiplicative Inverse Property?

As we are done with all the operations, this is a chance to summarize all of them, not just division.

	If B is…	**Then A + B will be …**		
A)	0	Greater than A	Less than A	Equal to A
B)	Positive	Greater than A	Less than A	Equal to A
C)	Negative	Greater than A	Less than A	Equal to A

	If B is…	**Then A – B will be …**		
D)	0	Greater than A	Less than A	Equal to A
E)	Positive	Greater than A	Less than A	Equal to A
F)	Negative	Greater than A	Less than A	Equal to A

	If the size of B is…	**Then A × B will be …**			
G)	0	Greater than A	Less than A	Equal to A	Equal to 0
H)	Between 0 and 1	Greater than A	Less than A	Equal to A	Equal to 0
I)	1	Greater than A	Less than A	Equal to A	Equal to 0
J)	Greater than 1	Greater than A	Less than A	Equal to A	Equal to 0

	If the size of B is…	**Then A ÷ B will be …**			
K)	0	Greater than A	Less than A	Equal to A	Undefined
L)	Between 0 and 1	Greater than A	Less than A	Equal to A	Undefined
M)	1	Greater than A	Less than A	Equal to A	Undefined
N)	Greater than 1	Greater than A	Less than A	Equal to A	Undefined

Concept Questions

When dealing with integers, the sign of the result depends on the sign of the starting numbers. The result could be always positive (P), always negative (N), or sometimes positive and sometimes negative (S). Label each of the following expressions as P, S, or N. If the answer is P or N, explain why. But if the answer is S, give one example that shows the product could be positive and one example that shows the product could be negative. *Note: some questions review previous topics as well as multiplication.*

Expression	Sign (circle one)	Examples or Explanation
pos ÷ pos	P S N	
pos ÷ neg	P S N	
neg ÷ neg	P S N	
neg ÷ pos	P S N	
pos × pos	P S N	
pos × neg	P S N	
neg × neg	P S N	
neg × pos	P S N	

Expression	Sign (circle one)	Examples or Explanation
pos + neg	P S N	
neg + pos	P S N	
neg + neg	P S N	
neg – pos	P S N	
neg – neg	P S N	
pos – pos	P S N	