Math Fundamentals for Statistics II (Math 95)

Unit 1: Finance

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“The ‘How’ and ‘Whys’ Guys”

2nd Edition (Summer 2016)

<u>1.0 Learning Cycle: Welcome to Math 95!</u>

We are happy to have you in this class and continuing your mathematical journey with us. Since this class is probably new to many of you, we have the following opening activity:

1. Meet 4 new people to the class and determine the following information:
    a. Name.
    b. When do they study math or what times <u>*could*</u> they study math.
    c. Something they are good at.
    d. How they got good/better at it.

Summarize your information in the following table:

Name	Study times/days	Good at	How?

Math 95 focuses on concepts over memorization, and in light of that, we allow 1 page of notes for each exam. We hope that you'll use this page, along with homework, notes, and review sections, to help you determine where you are at in the learning cycle and what you may need more practice with. In mathematics, it is important that you understand a learning cycle – let's see if we can come up with some good tips on how to learn a new concept or idea:

<u>LEARNING CYCLE TIPS/IDEAS:</u>

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<u>1.1: Starting Simple</u>

When it comes to money, it seems that everyone wants more of it. However, the way to make that happen is usually less clear. The problem of being able to buy something that we don't have the cash for currently is the basis for credit, and the idea of borrowing and lending has been around for nearly 4000 years. While in the early days, the monetary system was based on commodities like grain and livestock, it changed to be coins made of metal and then to paper money. Typically, the paper money was a certificate that was tied to a certain amount of precious metal (like silver or gold).

The United States moved to a gold standard in 1879, about 15 years after the civil war. The civil war produced problems with currency as both the North and South created their own. After re-unifying, the country moved to gold as a basis for currency and creditors could demand payment in gold based on their paper currency. The first major crisis was the Great Depression, causing the public to hoard gold as the way to build and maintain wealth. Great Britain moved away from the gold standard in 1931, and in 1933, President Roosevelt started the movement away from the gold standard. However, it took until 1971 for the US to completely abandon the gold standard.

Now picture this, you have a nice car and someone you don't know wants to borrow it for a while. Because it is your car, would you let them borrow it for free? Or would you charge them a little money to cover the fact that you aren't able to use the car when you want to. This idea of renting a car is similar to the idea of interest. Interest is like renting money for a while when you need more than you currently have. But what type of rent should be paid for money, and what would influence the amount of rent that should be charged?

Would you charge someone more rent for keeping the car longer or shorter? Would you charge someone more rent if the car was more expensive or if it was cheaper? And what rate would you charge – is it the same all the time or could the rates change from time to time? Would you charge more when there aren't as many cars to rent or maybe charge less when everyone is renting a car? Just like "surge" pricing with Uber! All of these concepts come into play when renting money too.

Now the value of our currency fluctuates with inflation and deflation. And along with these fluctuations go the interest rates – the amount charged to borrow money. The interest charged (like the rent for a car) is measured by three factors: the amount you borrow (**<u>principal</u>**), the length of time to repay (**<u>term</u>**), and the cost for borrowing as a percentage (**<u>interest rate</u>**). The only difference between borrowing and lending is who earns the interest!

Just like renting a car – if you rent it longer, it will cost you more. If you rent a fancier car that is more expensive, then it will cost you more. And if you're trying to rent a car when there aren't many because the supply is tight, then it will also cost you more. All three things come together to produce the amount of rent, and we need all three things in order to form the basis for our interest charge.

The most basic form of borrowing or investing is known as **<u>simple interest</u>**. In this type of interest, the interest rate is applied to the principal amount only. There is a formula associated with this type of growth called the **<u>simple interest formula</u>**: $I = P \cdot r \cdot t$. Here, the capital letters relate to monetary value (dollars, euro, yen, etc) and the lowercase letters relate to other quantities. ***<u>I</u>*** represents the amount of interest (earned or paid), ***<u>P</u>*** represents the amount of principal (invested or borrowed), ***<u>r</u>*** is the interest rate (as a decimal), and ***<u>t</u>*** is the time. Note that the interest rate needs to have a time associated with it and that quantity needs to match.

Example: Find the interest earned when investing…

A) \$5,000 at 6% annual simple interest for 7 years.

B) \$5,000 at 6% monthly simple interest for 7 yrs.

C) \$8,000 at 5% annual simple interest for 3 mths.

D) \$500 at 4% annual simple interest for 45 days.

Solution: The interest earned is…

A) $I = P \cdot r \cdot t = (5{,}000)(0.06)(7) = 2{,}100.$

B) The interest rate is monthly, so we need to measure time in months. 7 years is 84 months (each year is 12 months), so $I = P \cdot r \cdot t = (5{,}000)(0.06)(84) = 25{,}200.$

C) The interest rate is annual, so we need to measure time in years. 3 months is $\frac{3}{12}$ of a year, so

$I = P \cdot r \cdot t = (8{,}000)(0.05)\left(\frac{3}{12}\right) = 100.$

D) The interest rate is annual, so we need to measure time in years. 45 days is $\frac{45}{365}$ of a year, so

$I = P \cdot r \cdot t = (500)(0.04)\left(\frac{45}{365}\right) \approx 2.47.$

EXPLORE! Try some simple interest problems on your own. Focus on the 'time' units.

A) ** Find the interest paid on a loan for \$5,000 at 8% annual simple interest for 211 days.

B) Find the interest paid on a loan for \$500 at 8% annual simple interest for 2 years.

C) Find the interest earned on an investment of \$4,000 at 2% annual simple interest for 3 weeks.

D) Find the interest earned on an investment of \$800 at 9% monthly simple interest for 3 years.

In order to solve simple interest problems using $I = P \cdot r \cdot t$, we need to know three of the four pieces of information. In the problems above, we've know everything except the interest (*I*). Now we'll do an example showing how to calculate other items. Just like in Math 52, we can solve this equation for any of the missing pieces.

Example: A payday loan charges \$8.33 on a \$50 loan for 14 days. What is the annual simple interest rate?

Solution: We are missing something, but can use the formula to plug in everything else.
$I = P \cdot r \cdot t \Rightarrow 8.33 = 50 \cdot r \cdot \frac{14}{365}$. To solve for *r*, we will divide both sides by 50 and then divide by $\frac{14}{365}$.
$8.33 = 50 \cdot r \cdot \frac{14}{365} \Rightarrow \frac{8.33}{50} = r \cdot \frac{14}{365} \Rightarrow r = \frac{8.33}{50} \div \frac{14}{365} = 4.6042$. Make sure to report this rate as a percent, so the annual simple interest rate is 460.42%. WOW!

EXPLORE! Solve these simple interest problems for the missing piece.

A) ** Find the simple interest rate for a loan of 3 years requiring $500 interest on principal of $6,500.

B) Find the simple interest rate for a loan of 60 days requiring $50 interest on principal of $650.

C) Find the simple interest rate for a loan of 60 months with $2,350 interest on principal of $32,000.

Example: So far we have solved for *I* and *r*, so let's try a few for *P*. An investment guarantees interest of $600 on a simple annual interest rate of 10% over 2 years. How much do you need to invest?

Solution: We are missing something, but can use the formula to plug in everything else.
$I = P \cdot r \cdot t \Rightarrow 600 = P \cdot 0.10 \cdot 2$. To solve for *P*, we will divide both sides by 2 and then divide by 0.10.
$600 = P \cdot 0.10 \cdot 2 \Rightarrow \frac{600}{2} = P \cdot 0.10 \Rightarrow P = \frac{600}{2} \div 0.10 = 3{,}000$. We need to invest $3,000 in the account.

EXPLORE! Solve these simple interest problems for the missing piece.

A) ** Find the principal needed to earn $54 in interest on 6% annual simple interest over 3 years.

B) Find the principal needed to earn $27 in interest on 5% annual simple interest over 3 months.

C) Find the principal needed to earn $300 in interest on 4% annual simple interest over 45 days.

When considering simple interest as a way to save money, remember that the interest is earned on the principal only. We'll use a spreadsheet to calculate dollar amounts and match them below. As an example, we'll invest $1,000 into an account paying 12% annual interest. So each month the interest is 1%.

Month	Principal	Interest	Balance
0	1,000		
1	1,000	10	1,010
2	1,000	10	1,020
3	1,000	10	1,030
4	1,000	10	1,040
5	1,000	10	1,050

As you can see, the principal amount never changes, so the interest stays at $10 per month for the length of the loan. So the balance grows each month by $10, creating a sequence we've seen before in Math 52 – Arithmetic Sequences! If we were to graph our money over time, it would form a straight line making linear growth… and the slope of that line would be the rate of change ($10/mo).

We could create a new formula that would provide us with the future value of a simple interest investment, instead of just giving us the interest. The future value would include the principle and the interest, so the formula would be: $FV = P + I = P + P \cdot r \cdot t = P(1 + r \cdot t)$. Because it is money, FV will be in capital letters.

<u>Future value simple interest formula</u>: $FV = P(1 + r \cdot t)$.

So in the formula $FV = P(1 + r \cdot t)$, there is an ending amount and a starting amount. We've called the starting amount the principal, which is still appropriate. But there is another term that also works: **<u>present value</u>**. The present value is the amount of money needed today to generate a specific future value.

<u>Example:</u> What is the future value of $750 at 4% annual simple interest for 13 months?

<u>Solution:</u> Put in all the information into $FV = P(1 + r \cdot t)$. $FV = 750\left(1 + 0.04 \cdot \frac{13}{12}\right) = 782.50$.

Calculator note: *When putting in the information, you can put it all in with the parenthesis in one step, but if you prefer to go in steps, then compute what is inside the parenthesis first. Hit = and then multiply by the $750.*

<u>EXPLORE!</u> Determine the following:

A) ** Determine the future value of $8,000 at 3.2% annual simple interest for 5 years.

B) Determine the future value of $600 at 4.25% monthly simple interest for 10 months.

EXPLORE! Determine the following:

A) ** Determine the present value that will yield a future value of $2,400 at 3% annual simple interest for 2 years. (Calculate the parenthesis first, then divide).

B) Determine the present value that will yield a future value of $2,400 at 6% annual simple interest for 2 years.

C) If you want to purchase something that costs $800, how much do you need to invest at 4% annual simple interest for 1 year?

D) If we want to get $10,000 in 2 years, do we need to right now invest $10,000, more than $10,000 or less than $10,000? Explain.

The biggest applications of simple interest are loans and credit cards.

An **add-on interest** loan is a loan that computes all the interest up front and adds it to the total amount. The monthly payments are then computed by dividing this total (including interest) by the number of months.

Example: Francisco is looking to buy a new car for $28,000. After putting $1,000 down, he takes an add-on interest loan for the rest at 4% annual simple interest for 5 years. Find how much interest he will pay, and the monthly payment.

Solution: With $1,000 down, the principal amount is $27,000. We'll compute the interest and then add it on: $I = P \cdot r \cdot t \Rightarrow I = 27{,}000 \cdot 0.04 \cdot 5 = 5{,}400$. So Francisco will owe $5,400 in interest, but will also owe the $27,000 for a total amount of $32,400. The loan is for 5 years, which is 60 months, so his monthly payment will be $\frac{32{,}400}{60} = 540$. Francisco will pay $540 each month for 5 years.

<u>EXPLORE!</u> Determine the monthly payment and total interest paid for the following add-on interest loans.

A) ** The principal amount is $800 at 12% annual simple interest for 2 years.

B) The principal amount is $38,000 at 5.25% annual simple interest for 8 years.

C) The principal amount is $18,000 at 5.25% monthly simple interest for 2 years.

Credit cards use simple interest to calculate the **<u>finance charge</u>** each month. The finance charge (interest) is based on the **<u>average daily balance</u>** (**<u>ADB</u>** – principal), the interest rate, and the number of days in the month.

To help, it is good to remember the following about our calendar and the number of days in each.

January – 31	February – 28 (or 29)	March – 31	April – 30
May – 31	June – 30	July – 31	August – 31
September – 30	October – 31	November – 30	December – 31

<u>Example:</u> Compute the finance charge for an average daily balance of $517 at 17.99% annual simple interest during March.

<u>Solution:</u> $I = P \cdot r \cdot t \Rightarrow I = 517 \cdot 0.1799 \cdot \frac{31}{365} \approx 7.90$. The finance charge would be $7.90.

<u>EXPLORE!</u> Determine the finance charge for the following credit cards.

A) ** ADB is $1,100 at 19.99% annual simple interest over June.

B) ADB is $15,762 (average American credit card debt) at 14.9% annual simple interest (average American credit card rate) over October.

C) ADB is $152 at 17.99% annual simple interest over January. First – ballpark this and decide if you think the finance charge will be less than $5, between $5 and $15, or more than $15?

Credit cards can be complicated tools for many people, and some have sworn them off entirely as the risk outweighs the reward. Used properly, they can help build credit and are very convenient. Some credit card concepts are: new balance, **<u>minimum payment</u>** amount, finance charge, and remaining balance.

At the end of a billing cycle (usually a month), there is an amount still owed – this amount is your remaining balance. The credit card company computes your finance charge (and any other fees) and adds that to the remaining balance, creating the new balance. The new balance is used to compute the minimum payment amount which can vary between 3% and 5% of the new balance, rounded up to the nearest $5 and the payment must be at least $20. For these examples, we will use 4% as the rate required for the minimum payment amount. [Until 2004, the minimum was about 2% or less.]

<u>Example:</u> Petra has a credit card with an ADB of $684.28, with a remaining balance of $650.13. Her interest rate is 18.99% simple interest over November. Determine (A) her finance charge, (B) the new balance, and (C) the minimum payment amount.

<u>Solution:</u>

A) The finance charge is done as above: $I = P \cdot r \cdot t \Rightarrow I = 684.28 \cdot 0.1899 \cdot \frac{30}{365} \approx 10.68$ or $10.68.

B) The new balance is $650.13 + $10.68 = $660.81.

C) The minimum payment is $660.81(0.04) \approx 26.43$ which rounds to $30.

EXPLORE! Determine (i) the finance charge, (ii) the new balance, and (iii) the minimum payment amount.

A) Peter has a credit card with an ADB of $1,694.78, with a remaining balance of $1753.91. His interest rate is 13.99% simple interest over December.

B) Ophelia has a credit card with an ADB of $484.28, with a remaining balance of $390.25. Her interest rate is 24.99% simple interest over March.

C) The average American. ADB is $15,762 (average American credit card debt) at 14.9% annual simple interest (average American credit card rate) over August, with remaining balance of $15,800.

Group Activity with Computers:

Sophia must choose between keeping her current credit card with $2,500 balance and 14.99% simple annual interest or switching to a new card that came in the mail. The new card has 1.99% interest for 12 months, but requires a balance transfer fee of 3%. After the first year, the interest increases to 17.99%. Her minimum payment on the old card is $100 and she can't afford any higher payment. Should she transfer the money to the new card or stay with her current card? Explain your reasoning.

NOTE: *For many money concepts, there are tables of numbers to help calculate things quickly. Because simple interest is so quick, we don't actually use any tables for these problems!*

Formula Rolling Summary

At the end of each section in this unit, we will place the formulas you've seen in the section with other formulas from previous sections.

1.1: $I = P \cdot r \cdot t$ and $FV = P(1 + r \cdot t)$

Simple Interest and Future Value Simple Interest
Use for simple interest, add-on interest loans, and finding credit card finance charge (with average daily balance).
SOLVE FOR: Any variable.

EXPLORE! Determine when the formulas are appropriate **(without doing any computations)**:

A) When do you use $I = P \cdot r \cdot t$ compared to $FV = P(1 + r \cdot t)$?

B) How do you find the simple interest amount using $FV = P(1 + r \cdot t)$?

C) How do you find the future value with simple interest using $I = P \cdot r \cdot t$?

1.2: Compounding Problems

While simple interest is a nice way to work, it creates some problems. For example, if you invested money and earned some interest, it would be nice to be able to re-invest that interest into the account and continue earning interest. A simple interest account wouldn't allow this. Because if we allow this to happen, then we are earning interest on top of our interest and we call this compounding the interest as it grows more quickly.

First, let's review a bit with fractions.

<u>Interactive Example:</u> Determine the following values.

A) $0.03\left(\frac{1}{12}\right)$

B) $0.08\left(\frac{1}{4}\right)$

C) $\frac{0.03}{12}$

D) $\frac{0.08}{4}$

How could you rewrite $0.07\left(\frac{1}{12}\right)$ so that it is simpler to calculate?

Since we often have time of 1 month or 1 quarter or 1 year, the numerator will be 1 in the fraction representing time. This technique of taking the rate and the time and combining them into one fraction is known as the **<u>periodic rate</u>**. The periodic rate takes an annual rate and computes what portion you'll earn for a given time period.

<u>Example:</u> Find the periodic rate for these rate and time values.

A) 6% annual interest, monthly

B) 40% annual interest, quarterly

C) 8% annual interest, semiannually

D) 20% annual interest, annually

<u>Solutions:</u>

A) $0.06\left(\frac{1}{12}\right)=\frac{0.06}{12}=0.005$

B) $0.40\left(\frac{1}{4}\right)=\frac{0.40}{4}=0.10$

C) $0.08\left(\frac{1}{2}\right)=\frac{0.08}{2}=0.04$

D) $0.20\left(\frac{1}{1}\right)=0.20$

The periodic rate is often simplified to the symbol *i*, and uses the following form: $i=\frac{r}{n}$, where *i* is the periodic rate, *r* is the annual interest rate, and *n* is the number of compounding periods per year.

This table may be helpful during the homework and these are the most common periods:

Time Measurement	# of Compounding Periods		Time Measurement	# of Compounding Periods
Annually	1		Semi-Annually	2
Quarterly	4		Monthly	12
Semi-Monthly	24		Bi-weekly	26
Weekly	52		Daily	365

Example: Let's do a sample problem by investing \$10,000 at 6% annual interest, but see how things work month by month where the interest we earn rolls into the principal.

Solution:

i) Month #1: $FV = P(1 + r \cdot t) = 10{,}000\left(1 + 0.06\left(\frac{1}{12}\right)\right) = 10{,}000\left(1 + \frac{0.06}{12}\right) = 10{,}050$.

ii) Month #2: $FV = P(1 + r \cdot t) = 10{,}050\left(1 + 0.06\left(\frac{1}{12}\right)\right) = 10{,}050\left(1 + \frac{0.06}{12}\right) = 10{,}100.25$

iii) Month #3: $FV = P(1 + r \cdot t) = 10{,}100.25\left(1 + 0.06\left(\frac{1}{12}\right)\right) = 10{,}100.25\left(1 + \frac{0.06}{12}\right) = 10{,}150.75125$

We would probably round these to the nearest penny, but the whole amount does exist in the calculation. Rounding it would show about \$10,150.75.

So the first month, we earned \$50 in interest. The second month, we earned \$50.25, with the extra 25¢ because we earned interest on the extra \$50 that was in the account. The third month we earned more than \$50.50 because of the compounding interest – interest earned on the interest.

Each time we seem to multiply the previous amount by $\left(1 + \frac{0.06}{12}\right)$ which is $(1 + i)$ using periodic rate. If we wanted to save time, we could multiply repeatedly by this $(1 + i)$ instead of needing to do the individual steps. What a time saver! How could we represent $(1+i)\,(1+i)\,(1+i)\,(1+i)\,(1+i)$ in a simpler way?

$(1+i)\,(1+i)\,(1+i)\,(1+i)\,(1+i) = (1+i)^5$ which is much easier to write and use. This is the basis for the formulas we're about to see. We repeatedly multiply by this quantity until we've reached the appropriate number of time periods. For compound interest accounts, the formula can be written two ways (one standard and one periodic): $FV = P\left(1 + \frac{r}{n}\right)^{nt}$ or $FV = P(1+i)^{nt}$. Here *t* is the time in years and *n* is the number of compounding periods per year.

Interactive Example: Find the future value of \$10,000 at 6% annual interest compounded monthly after: (A) 3 months, (B) 1 year, (C) 10 years, (D) 30 years. Compare each with the value using simple interest. Round to the nearest penny at the *end* of the computations.

Solutions:

A) $FV = P(1+i)^{nt} = 10{,}000\left(1 + \frac{0.06}{12}\right)^{12\left(\frac{3}{12}\right)} = 10{,}000\left(1 + \frac{0.06}{12}\right)^{3} = 10{,}150.75125 \approx 10{,}150.75$

Simple Interest: $FV = 10{,}000\left(1 + 0.06\left(\frac{3}{12}\right)\right) = 10{,}150$. Compound interest better by 75 ¢.

B) $FV = P(1+i)^{nt} = 10{,}000\left(1 + \frac{0.06}{12}\right)^{12(1)} = 10{,}000\left(1 + \frac{0.06}{12}\right)^{12} =$

Simple Interest: $FV = 10{,}000(1 + 0.06(\quad)) =$

C) $FV = P(1+i)^{nt} = 10{,}000\left(1 + \frac{0.06}{12}\right)^{12(10)} =$

Simple Interest: $FV = 10{,}000(1 + 0.06(\quad)) =$

D) $FV = P(1+i)^{nt} =$

Simple Interest: $FV = 10{,}000(1 + 0.06(\quad)) =$

EXPLORE! What would you say about comparing simple interest and compound interest accounts…

A) over the short term?

B) over the long term?

This idea is known as the Time-Value of Money; the idea that compounding the interest will allow for massive growth over longer periods of time, but will be very close to simple interest in the short term. We saw this in Math 52 (Unit 1) – *Arithmetic sequences* formed straight lines that were increasing or decreasing. But *geometric sequences* formed curves that would grow much faster after a certain period of time. [Think about the allowances problem with the two girls where one got a penny on the first day and it doubled each day!]

$FV = P(1+i)^{nt}$ is an equation that takes a lump sum of money and keeps it in an account for a fixed amount of time at a fixed interest rate. For what we will do in this class, that means we can solve for *FV* or for *P*.

Example: Find the future value of $8,000 at 3.75% annual interest compounded quarterly for 5 years, and determine how much interest was earned.

Solution: $FV = P(1+i)^{nt} \Rightarrow FV = 8{,}000\left(1+\frac{0.0375}{4}\right)^{4(5)} = 8{,}000\left(1+\frac{0.0375}{4}\right)^{20} \approx 9{,}641.42$. The future value is $9,641.42. This future value includes our initial $8,000 and by removing it, we'll see the interest by itself. The interest is $9,641.42 – $8,000 = $1,641.42.

EXPLORE! Try some on your own!

	If the future value is	Then the present value should be
A)	$23,000	More than $23,000 Less than $23,000 Equal to $23,000
	If the present value is	Then the future value should be
B)	$7,000	More than $7,000 Less than $7,000 Equal to $7,000

If you know the future value and are looking for the present value, you can use calculator in a few ways.

- Method 1: Store the value for later. Press the "STO" button and select a memory location. Then type in the future value and press ÷ then "RCL" and your memory location.
- Method 2: Type in the information next to *P* in the formula, then press x^{-1} and multiply by the *FV*.

EXPLORE! Try some on your own!

A) Determine the future value of $5,000 at 2.99% annual interest compounded weekly for 10 years, and determine the amount of interest earned.

B) Determine the future value of $4,500 at 7.49% annual interest compounded monthly for 13 years, and determine the amount of interest earned.

C) Determine the *present value* of $9,000 at 5% annual interest compounded daily for 7 years, and determine the amount of interest earned.

D) Determine the *present value* of $750,000 at 5.6% annual interest compounded monthly for 27 years, and determine the amount of interest earned.

E) If you were 27 years from retirement and knew that you needed $750,000 at retirement to be able to fully retire and have your expenses covered, how much cash would you need right now (at 5.6% annual interest compounded monthly)? Do you have this kind of money? (Seriously – you could ask your parents and friends to chip in… right?)

One last application of compound interest is in the idea of price inflation. Over time, prices tend to increase and often, inflation is given a factor as a percentage known as the **inflation rate**. Historically, the US has averaged inflation between 1% and 3%, but it has peaked in some months and been negative in others, which is called deflation.
http://www.usinflationcalculator.com/inflation/historical-inflation-rates/

Example: Using the 2% inflation as a guide, determine how much a gallon of milk will cost 20 years from now (current price in May 2016 was $2.99/gallon).

Solution: Thankfully, we can use our compound formula because the inflation grows each year on the previous amount, not on some arbitrary base. The 2% is an annual amount and it compounds annually (for our example): $FV = P(1+i)^{nt} \Rightarrow FV = 2.99\left(1+\frac{0.02}{1}\right)^{1(20)} = 2.99(1+0.02)^{20} \approx 4.44$. So a gallon of milk 20 years from now will cost $4.44.

EXPLORE! Try out a few more problems using inflation.

A) The long term average inflation rate for health care prices is 5.42% annually. If a health care plan costs $800 per month now…
   i. What is the annual health care cost now?

   ii. What is the annual health care cost 15 years from now?

   iii. What would be the monthly health care cost 15 years from now?

B) Wages and salaries are rising at about 1.3% annually. If a job has a salary of $85,000 per year now, how much would that salary be 18 years from now?

C) Assuming that inflation is 3% annually, and you currently pay $3.19 for gas…
   i. How much will you pay for gas 30 years from now?

   ii. How much would you have paid for gas 30 years ago? (Hint – put in a negative exponent!)

D) Tuition has been rising at about 8% annually for the past 30 years. If this continues, how much would the tuition be at USC in 20 years (currently $50,348 per year in May 2016)?

So there are formulas for these as we've seen, but there are other ways to compute that are sometimes used for speed. These tables are able to compute the challenging part (with the exponents) and all you need to do is multiply by the principal!

If you invest $1 at 4% annual interest compounded annually for 8 years, you could use the formula $FV = P(1+i)^{nt} \Rightarrow FV = 1\left(1+\frac{0.04}{1}\right)^{1(8)} = 1(1+0.04)^{8} \approx 1.3685690504052736$ or about 1.3686. Find this number in the table by looking up 4% and 8 periods.

TABLE 1 (Part 1 of 2)

(FVIF – Future Value Interest Factor for 1% to 7%)

Periods	1%	2%	3%	4%	5%	6%	7%
1	1.0100	1.0200	1.0300	1.0400	1.0500	1.0600	1.0700
2	1.0201	1.0404	1.0609	1.0816	1.1025	1.1236	1.1449
3	1.0303	1.0612	1.0927	1.1249	1.1576	1.1910	1.2250
4	1.0406	1.0824	1.1255	1.1699	1.2155	1.2625	1.3108
5	1.0510	1.1041	1.1593	1.2167	1.2763	1.3382	1.4026
6	1.0615	1.1262	1.1941	1.2653	1.3401	1.4185	1.5007
7	1.0721	1.1487	1.2299	1.3159	1.4071	1.5036	1.6058
8	1.0829	1.1717	1.2668	**1.3686**	1.4775	1.5938	1.7182
9	1.0937	1.1951	1.3048	1.4233	1.5513	1.6895	1.8385
10	1.1046	1.2190	1.3439	1.4802	1.6289	1.7908	1.9672
11	1.1157	1.2434	1.3842	1.5395	1.7103	1.8983	2.1049
12	1.1268	1.2682	1.4258	1.6010	1.7959	2.0122	2.2522
13	1.1381	1.2936	1.4685	1.6651	1.8856	2.1329	2.4098
14	1.1495	1.3195	1.5126	1.7317	1.9799	2.2609	2.5785
15	1.1610	1.3459	1.5580	1.8009	2.0789	2.3966	2.7590
16	1.1726	1.3728	1.6047	1.8730	2.1829	2.5404	2.9522
17	1.1843	1.4002	1.6528	1.9479	2.2920	2.6928	3.1588
18	1.1961	1.4282	1.7024	2.0258	2.4066	2.8543	3.3799
19	1.2081	1.4568	1.7535	2.1068	2.5270	3.0256	3.6165
20	1.2202	1.4859	1.8061	2.1911	2.6533	3.2071	3.8697
21	1.2324	1.5157	1.8603	2.2788	2.7860	3.3996	4.1406
22	1.2447	1.5460	1.9161	2.3699	2.9253	3.6035	4.4304
23	1.2572	1.5769	1.9736	2.4647	3.0715	3.8197	4.7405
24	1.2697	1.6084	2.0328	2.5633	3.2251	4.0489	5.0724
25	1.2824	1.6406	2.0938	2.6658	3.3864	4.2919	5.4274
26	1.2953	1.6734	2.1566	2.7725	3.5557	4.5494	5.8074
27	1.3082	1.7069	2.2213	2.8834	3.7335	4.8223	6.2139
28	1.3213	1.7410	2.2879	2.9987	3.9201	5.1117	6.6488
29	1.3345	1.7758	2.3566	3.1187	4.1161	5.4184	7.1143
30	1.3478	1.8114	2.4273	3.2434	4.3219	5.7435	7.6123
31	1.3613	1.8476	2.5001	3.3731	4.5380	6.0881	8.1451
32	1.3749	1.8845	2.5751	3.5081	4.7649	6.4534	8.7153
33	1.3887	1.9222	2.6523	3.6484	5.0032	6.8406	9.3253
34	1.4026	1.9607	2.7319	3.7943	5.2533	7.2510	9.9781
35	1.4166	1.9999	2.8139	3.9461	5.5160	7.6861	10.6766
36	1.4308	2.0399	2.8983	4.1039	5.7918	8.1473	11.4239
37	1.4451	2.0807	2.9852	4.2681	6.0814	8.6361	12.2236
38	1.4595	2.1223	3.0748	4.4388	6.3855	9.1543	13.0793
39	1.4741	2.1647	3.1670	4.6164	6.7048	9.7035	13.9948
40	1.4889	2.2080	3.2620	4.8010	7.0400	10.2857	14.9745

TABLE 1 (Part 2 of 2)

(FVIF for 8% to 14%)

Periods	8%	9%	10%	11%	12%	13%	14%
1	1.0800	1.0900	1.1000	1.1100	1.1200	1.1300	1.1400
2	1.1664	1.1881	1.2100	1.2321	1.2544	1.2769	1.2996
3	1.2597	1.2950	1.3310	1.3676	1.4049	1.4429	1.4815
4	1.3605	1.4116	1.4641	1.5181	1.5735	1.6305	1.6890
5	1.4693	1.5386	1.6105	1.6851	1.7623	1.8424	1.9254
6	1.5869	1.6771	1.7716	1.8704	1.9738	2.0820	2.1950
7	1.7138	1.8280	1.9487	2.0762	2.2107	2.3526	2.5023
8	1.8509	1.9926	2.1436	2.3045	2.4760	2.6584	2.8526
9	1.9990	2.1719	2.3579	2.5580	2.7731	3.0040	3.2519
10	2.1589	2.3674	2.5937	2.8394	3.1058	3.3946	3.7072
11	2.3316	2.5804	2.8531	3.1518	3.4785	3.8359	4.2262
12	2.5182	2.8127	3.1384	3.4985	3.8960	4.3345	4.8179
13	2.7196	3.0658	3.4523	3.8833	4.3635	4.8980	5.4924
14	2.9372	3.3417	3.7975	4.3104	4.8871	5.5348	6.2613
15	3.1722	3.6425	4.1772	4.7846	5.4736	6.2543	7.1379
16	3.4259	3.9703	4.5950	5.3109	6.1304	7.0673	8.1372
17	3.7000	4.3276	5.0545	5.8951	6.8660	7.9861	9.2765
18	3.9960	4.7171	5.5599	6.5436	7.6900	9.0243	10.5752
19	4.3157	5.1417	6.1159	7.2633	8.6128	10.1974	12.0557
20	4.6610	5.6044	6.7275	8.0623	9.6463	11.5231	13.7435
21	5.0338	6.1088	7.4002	8.9492	10.8038	13.0211	15.6676
22	5.4365	6.6586	8.1403	9.9336	12.1003	14.7138	17.8610
23	5.8715	7.2579	8.9543	11.0263	13.5523	16.6266	20.3616
24	6.3412	7.9111	9.8497	12.2392	15.1786	18.7881	23.2122
25	6.8485	8.6231	10.8347	13.5855	17.0001	21.2305	26.4619
26	7.3964	9.3992	11.9182	15.0799	19.0401	23.9905	30.1666
27	7.9881	10.2451	13.1100	16.7386	21.3249	27.1093	34.3899
28	8.6271	11.1671	14.4210	18.5799	23.8839	30.6335	39.2045
29	9.3173	12.1722	15.8631	20.6237	26.7499	34.6158	44.6931
30	10.0627	13.2677	17.4494	22.8923	29.9599	39.1159	50.9502
31	10.8677	14.4618	19.1943	25.4104	33.5551	44.2010	58.0832
32	11.7371	15.7633	21.1138	28.2056	37.5817	49.9471	66.2148
33	12.6760	17.1820	23.2252	31.3082	42.0915	56.4402	75.4849
34	13.6901	18.7284	25.5477	34.7521	47.1425	63.7774	86.0528
35	14.7853	20.4140	28.1024	38.5749	52.7996	72.0685	98.1002
36	15.9682	22.2512	30.9127	42.8181	59.1356	81.4374	111.8342
37	17.2456	24.2538	34.0039	47.5281	66.2318	92.0243	127.4910
38	18.6253	26.4367	37.4043	52.7562	74.1797	103.9874	145.3397
39	20.1153	28.8160	41.1448	58.5593	83.0812	117.5058	165.6873
40	21.7245	31.4094	45.2593	65.0009	93.0510	132.7816	188.8835

Without a calculator, you could use these tables to get values that are very close to what a calculator could get and you'd only need to multiply numbers (no exponents). A very basic calculator could do this quickly. Using the table takes a little bit of time, but can be helpful too.

Oh, and the FVIF stands for **Future Value Interest Factor**, meaning the number you multiply the principal by!

EXPLORE! Using the tables, compute the following. Find the…

A) ** future value of investing $4,000 at 5% annual interest compounded annually for 25 years.

B) ** future value of investing $8,000 at 12% annual interest compounded monthly for 3 years.

C) future value of investing $5,800 at 6% annual simple interest compounded semi-annually for 6 years.

D) future value of investing $11,900 at 13% annual simple interest compounded annually for 23 years.

E) future value of investing $2,500 at 28% annual simple interest compounded quarterly for 9 years.

EXPLORE (2)! Using the formula $FV = P(1+i)^{nt}$, confirm the information above.

A)

B)

C)

D)

E)

F) The tables are interesting and sometimes quicker… but do you see a drawback to the tables? Explain.

Formula Rolling Summary

1.1: $I = P \cdot r \cdot t$ and $FV = P(1 + r \cdot t)$

Simple Interest and Future Value Simple Interest
Use for simple interest, add-on interest loans, and finding credit card finance charge (with average daily balance).
SOLVE FOR: Any variable.

1.2: $FV = P(1+i)^{nt}$

Future Value Compound Interest
Use for lump sum compound interest investments, inflation (finding price of something in the future or past)
SOLVE FOR: *FV* or *P* only.

EXPLORE! Determine when the formulas are appropriate **(without doing any computations)**:

A) When do you use $FV = P(1+i)^{nt}$ compared to $FV = P(1 + r \cdot t)$?

B) If you invest at 5% annual interest compounded monthly and your friend, Steve, invests the same amount at 5% annual simple interest, who will have more in 2 years?

C) Which of these formulas are for lump sum investments?

D) Can you use $FV = P(1 + r \cdot t)$ to calculate the price of an item with inflation of 4% per year? Why or why not?

E) Can we use $FV = P(1+i)^{nt}$ to calculate the amount of interest? If so, how? If not, explain why not.

1.3: A Better Way to Save - Annuities

So far we have seen ways to save money with simple or compound interest. However, each of those methods required a lump-sum deposit. If we want to reach a large end goal but don't have the initial cash for a lump-sum deposit, perhaps investing smaller amounts repeatedly could get to the same goal.

Example: If you want to have $500,000 for retirement in 25 years, how much do you need to invest now at 5.75% annual interest compounded monthly?

Solution: This uses the Future Value (Compound Interest), so we can plug in all the relevant information:

$$FV = P(1+i)^{nt} \Rightarrow 500{,}000 = P\left(1+\tfrac{0.0575}{12}\right)^{12(25)} \Rightarrow 500{,}000 = P\left(1+\tfrac{0.0575}{12}\right)^{300} \Rightarrow P = 119{,}168.82$$

You could check this to see if it works too:

$$FV = P(1+i)^{nt} \Rightarrow FV = 119{,}168.82\left(1+\tfrac{0.0575}{12}\right)^{300} \Rightarrow FV \approx 499{,}999.9983$$ which rounds to $500,000. ☺

Of course, most people don't have $120,000 right now to save for retirement in one fail swoop! Instead, we may need to invest with smaller amounts over a period of time.

Growing up, my mother was extremely bad with her money. Knowing this, she would invest in a local "Christmas" club. Basically, some of her money would be put into the bank each month until on December 1st, then the bank would put the money back in her account and she could buy presents. Yup. That really happened.

We can practice with this type of account and see how it works.

Example: If my mom signs up for a Christmas Club in August and invests $100 each month starting at the end of the month. Her interest rate was 12% annual interest compounded monthly, see how much she gets paid on December first.

Solution: We'll take each investment and use our formula to find the value of each part, then add them up.

	August	September	October	November		Totals
Investment	$100	$100	$100	$100		$400
# of Months invested	3	2	1	0		
Future Value	$103.03	$102.01	$101.00	$100		$406.04

Because she didn't invest very long, there wasn't much interest to earn from this. Sorry mom.

EXPLORE! But was there still a benefit to this type of account? Explain why or why not.

While this is fairly interesting, there is a better way. For years, people who were very good at computation would make tables of figures. They were called… you'll never believe this… computers! Before 1935, here's the definition of **computer** from the dictionary: "A person who performs arithmetic calculations."

The next few tables help with just such calculations – they are titled FVIFA or Future Value Interest Factor Annuity. An **annuity** is a set of fixed payments made from an account or into an account. My mom made annuity payments, even if it was for a short while, into her account. Took the money out of her hands so she couldn't spend it until later and we were all able to get new socks and underwear (and calculators!).

TABLE 2 (Part 1 of 2)

(FVIFA – Future Value Interest Factor Annuity for 1% to 7%)

Periods	1%	2%	3%	4%	5%	6%	7%
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	2.0100	2.0200	2.0300	2.0400	2.0500	2.0600	2.0700
3	3.0301	3.0604	3.0909	3.1216	3.1525	3.1836	3.2149
4	4.0604	4.1216	4.1836	4.2465	4.3101	4.3746	4.4399
5	5.1010	5.2040	5.3091	5.4163	5.5256	5.6371	5.7507
6	6.1520	6.3081	6.4684	6.6330	6.8019	6.9753	7.1533
7	7.2135	7.4343	7.6625	7.8983	8.1420	8.3938	8.6540
8	8.2857	8.5830	8.8923	9.2142	9.5491	9.8975	10.2598
9	9.3685	9.7546	10.1591	10.5828	11.0266	11.4913	11.9780
10	10.4622	10.9497	11.4639	12.0061	12.5779	13.1808	13.8164
11	11.5668	12.1687	12.8078	13.4864	14.2068	14.9716	15.7836
12	12.6825	13.4121	14.1920	15.0258	15.9171	16.8699	17.8885
13	13.8093	14.6803	15.6178	16.6268	17.7130	18.8821	20.1406
14	14.9474	15.9739	17.0863	18.2919	19.5986	21.0151	22.5505
15	16.0969	17.2934	18.5989	20.0236	21.5786	23.2760	25.1290
16	17.2579	18.6393	20.1569	21.8245	23.6575	25.6725	27.8881
17	18.4304	20.0121	21.7616	23.6975	25.8404	28.2129	30.8402
18	19.6147	21.4123	23.4144	25.6454	28.1324	30.9057	33.9990
19	20.8109	22.8406	25.1169	27.6712	30.5390	33.7600	37.3790
20	22.0190	24.2974	26.8704	29.7781	33.0660	36.7856	40.9955
21	23.2392	25.7833	28.6765	31.9692	35.7193	39.9927	44.8652
22	24.4716	27.2990	30.5368	34.2480	38.5052	43.3923	49.0057
23	25.7163	28.8450	32.4529	36.6179	41.4305	46.9958	53.4361
24	26.9735	30.4219	34.4265	39.0826	44.5020	50.8156	58.1767
25	28.2432	32.0303	36.4593	41.6459	47.7271	54.8645	63.2490
26	29.5256	33.6709	38.5530	44.3117	51.1135	59.1564	68.6765
27	30.8209	35.3443	40.7096	47.0842	54.6691	63.7058	74.4838
28	32.1291	37.0512	42.9309	49.9676	58.4026	68.5281	80.6977
29	33.4504	38.7922	45.2189	52.9663	62.3227	73.6398	87.3465
30	34.7849	40.5681	47.5754	56.0849	66.4388	79.0582	94.4608
31	36.1327	42.3794	50.0027	59.3283	70.7608	84.8017	102.0730
32	37.4941	44.2270	52.5028	62.7015	75.2988	90.8898	110.2182
33	38.8690	46.1116	55.0778	66.2095	80.0638	97.3432	118.9334
34	40.2577	48.0338	57.7302	69.8579	85.0670	104.1838	128.2588
35	41.6603	49.9945	60.4621	73.6522	90.3203	111.4348	138.2369
36	43.0769	51.9944	63.2759	77.5983	95.8363	119.1209	148.9135
37	44.5076	54.0343	66.1742	81.7022	101.6281	127.2681	160.3374
38	45.9527	56.1149	69.1594	85.9703	107.7095	135.9042	172.5610
39	47.4123	58.2372	72.2342	90.4091	114.0950	145.0585	185.6403
40	48.8864	60.4020	75.4013	95.0255	120.7998	154.7620	199.6351

TABLE 2 (Part 2 of 2)

(FVIFA for 8% to 14%)

Periods	8%	9%	10%	11%	12%	13%	14%
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	2.0800	2.0900	2.1000	2.1100	2.1200	2.1300	2.1400
3	3.2464	3.2781	3.3100	3.3421	3.3744	3.4069	3.4396
4	4.5061	4.5731	4.6410	4.7097	4.7793	4.8498	4.9211
5	5.8666	5.9847	6.1051	6.2278	6.3528	6.4803	6.6101
6	7.3359	7.5233	7.7156	7.9129	8.1152	8.3227	8.5355
7	8.9228	9.2004	9.4872	9.7833	10.0890	10.4047	10.7305
8	10.6366	11.0285	11.4359	11.8594	12.2997	12.7573	13.2328
9	12.4876	13.0210	13.5795	14.1640	14.7757	15.4157	16.0853
10	14.4866	15.1929	15.9374	16.7220	17.5487	18.4197	19.3373
11	16.6455	17.5603	18.5312	19.5614	20.6546	21.8143	23.0445
12	18.9771	20.1407	21.3843	22.7132	24.1331	25.6502	27.2707
13	21.4953	22.9534	24.5227	26.2116	28.0291	29.9847	32.0887
14	24.2149	26.0192	27.9750	30.0949	32.3926	34.8827	37.5811
15	27.1521	29.3609	31.7725	34.4054	37.2797	40.4175	43.8424
16	30.3243	33.0034	35.9497	39.1899	42.7533	46.6717	50.9804
17	33.7502	36.9737	40.5447	44.5008	48.8837	53.7391	59.1176
18	37.4502	41.3013	45.5992	50.3959	55.7497	61.7251	68.3941
19	41.4463	46.0185	51.1591	56.9395	63.4397	70.7494	78.9692
20	45.7620	51.1601	57.2750	64.2028	72.0524	80.9468	91.0249
21	50.4229	56.7645	64.0025	72.2651	81.6987	92.4699	104.7684
22	55.4568	62.8733	71.4027	81.2143	92.5026	105.4910	120.4360
23	60.8933	69.5319	79.5430	91.1479	104.6029	120.2048	138.2970
24	66.7648	76.7898	88.4973	102.1742	118.1552	136.8315	158.6586
25	73.1059	84.7009	98.3471	114.4133	133.3339	155.6196	181.8708
26	79.9544	93.3240	109.1818	127.9988	150.3339	176.8501	208.3327
27	87.3508	102.7231	121.0999	143.0786	169.3740	200.8406	238.4993
28	95.3388	112.9682	134.2099	159.8173	190.6989	227.9499	272.8892
29	103.9659	124.1354	148.6309	178.3972	214.5828	258.5834	312.0937
30	113.2832	136.3075	164.4940	199.0209	241.3327	293.1992	356.7868
31	123.3459	149.5752	181.9434	221.9132	271.2926	332.3151	407.7370
32	134.2135	164.0370	201.1378	247.3236	304.8477	376.5161	465.8202
33	145.9506	179.8003	222.2515	275.5292	342.4294	426.4632	532.0350
34	158.6267	196.9823	245.4767	306.8374	384.5210	482.9034	607.5199
35	172.3168	215.7108	271.0244	341.5896	431.6635	546.6808	693.5727
36	187.1021	236.1247	299.1268	380.1644	484.4631	618.7493	791.6729
37	203.0703	258.3759	330.0395	422.9825	543.5987	700.1867	903.5071
38	220.3159	282.6298	364.0434	470.5106	609.8305	792.2110	1030.9981
39	238.9412	309.0665	401.4478	523.2667	684.0102	896.1984	1176.3378
40	259.0565	337.8824	442.5926	581.8261	767.0914	1013.7042	1342.0251

Example: If we invest $100 each month and earn 12% annual interest compounded monthly, how much would we have in 4 months?

Solution: Since the interest is compounded monthly, find the periodic rate: 12% ÷ 12 = 1%. Look up 1% and 4 periods (months) to find the FVIFA of 4.0604. The future value is: FV = $100(4.0604) = $406.04. That's the same amount my mom had and we didn't have to do 4 separate calculations!

EXPLORE! Try some more with FVIFA tables.

A) ** If you invest $5,500 each year in an account earning 8% annual interest compounded annually, how much would you have in 23 years? [$5,500 is the max annual contribution to a Roth IRA.]

B) If you invest $18,000 each year in an account earning 7% compounded annually, how much would you have in 23 years? [$18,000 is the max annual contribution to a 401k retirement account.]

C) If you invest $50 each month into an account earning 5% annual interest compounded annually, how much would you have in 15 years? [Hint: make the $50 per month into an annual payment]

D) According to Edmunds.com, the average monthly payment for a new vehicle is $479 per month for 60 months (5 years). Instead of buying the car, if you invest $479 each month at 9% annual interest compounded annually, how much would you have in 5 years?

E) What is a drawback of the table?

The table is pretty nice, but how would we calculate 4.25% interest… or what about doing it for 30 years or 40 years? This is a bit of a problem. Thankfully, there is a formula to help out. Based on the sum of a geometric sequence, we can adapt the compound interest formula to something easier to use:

$$FV = pmt + pmt(1+i) + pmt(1+i)^2 + pmt(1+i)^3 + ...pmt(1+i)^{nt-1} \Rightarrow FV = pmt\frac{\left((1+i)^{nt}-1\right)}{i}$$

Let's practice this one as it does look a bit more complicated than just a lump sum investment.

Example: Find the future value of investing $5,500 at 8% compounded annually for 23 years.

Solution: Plug in the pieces into the formula, remember that the periodic rate is 8% ÷ 1 = 8%.

$$FV = pmt\frac{\left((1+i)^{nt}-1\right)}{i} \Rightarrow FV = 5{,}500\frac{\left((1+0.08)^{23}-1\right)}{0.08} \Rightarrow FV = 5{,}500(60.89329557...) \approx \$334{,}913.13$$

NOTE: compute the portions in parenthesis first, then finally multiply by 5,500.

Example: If you invest $50 each month into an account earning 5% annual interest compounded monthly, how much would you have in 15 years? How much interest did you earn?

Solution: Plug the pieces into the formula and off we go!

$$FV = pmt\frac{\left((1+i)^{nt}-1\right)}{i} \Rightarrow FV = 50\frac{\left(\left(1+\frac{0.05}{12}\right)^{180}-1\right)}{\left(\frac{0.05}{12}\right)} \Rightarrow FV = 50(267.2889437...) \approx \$13,364.45$$

So after 15 years, we've got over $13,000. But a lot of that is our money, right? We invested $50 each month into the account, and over 15 years, 180 months, that adds up to be $50(180) = $9,000. So the amount of interest must be $13,364.45 – 9,000 = $4,364.45.

EXPLORE! Try some investing on your own using the formula.

A) If you invest 10% of your income and make $5,000 each month, how much will you have in 30 years if you invest at 6% annual interest compounded monthly? Also, determine how much interest was earned.

B) If you invest 10% of your income and make $5,000 each month, how much will you have in 25 years if you invest at 6% annual interest compounded monthly? Also, determine how much interest was earned.

C) If you invest 10% of your income and make $5,000 each month, how much will you have in 20 years if you invest at 6% annual interest compounded monthly? Also, determine how much interest was earned.

D) If you invest 5% of your income and make $5,000 each month, how much will you have in 30 years if you invest at 6% annual interest compounded monthly? Also, determine the interest earned.

E) Analyze the difference between investing less for longer, and waiting to invest.

EXPLORE! Invest with Annuities a few more times.

A) ** If you invest $5,500 each year for a Roth IRA, and you do this for 25 years, how much will be in the account if it earns 7% annual interest compounded annually. (use the table AND the formula – this assumes you start investing at age 40 and finish at age 65 so you can retire) Figure out how much you paid in and how much interest was earned.

B) Someone who is over 50 years old can "catch-up" with extra money into the Roth. If you invest $6,500 each year for a Roth IRA, and you do this for 15 years (age 50 to 65 only), how much will be in the account if it earns 7% annual interest compounded annually. (use the table AND the formula). Figure out how much you paid in and how much interest was earned.

C) If you invest $5,500 each year for a Roth IRA, and you do this for 10 years (from age 25 to age 35), then stop investing completely. How much would you have at age 65 when you retire? Assume 7% annual interest compounded annually. Determine how much you paid in and the amount of interest.

D) If you invested $5,500 each year from age 25 to age 50, then invested $6,500 from age 50 to age 65, determine how much money you would have in the account. ? Assume 7% annual interest compounded annually. Determine how much you paid in and the amount of interest.

The Roth IRA is an exceptional tool as all interest earned is Tax Free!

<u>EXPLORE!</u> Lastly, we'll work annuities backwards – we know how much we want to have but not sure how much to put away each month.

A) ** How much money do you need to invest each month at 5% annual interest compounded monthly for 18 years to end with $50,000 (for your baby's education)? Determine the interest you earned.

B) How much money do you need to invest each month at 6% annual interest compounded monthly for 35 years to end with $1,000,000 (for your retirement)? Determine how much interest you earned.

C) **(L)** Because of extraordinary circumstances, your friend is now 50 years old and hasn't started saving for retirement yet. She wants to have $250,000 when she retires at age 70. How much should she invest each month at 6% annual interest compounded monthly? Determine the interest she earned.

D) **(R)** Because of extraordinary circumstances, your friend is now 40 years old and hasn't started saving for retirement yet. She wants to have $250,000 when she retires at age 70. How much should she invest each month at 6% annual interest compounded monthly? Determine the interest she earned.

E) Based on (C) and (D) above, what if she started investing at age 25 until age 70, how much should she invest each month at 6% annual interest compounded monthly to end with the $250,000?

F) Why can we not use the table for these computations?

<u>Formula Rolling Summary</u>

1.1: $I = P \cdot r \cdot t$ and $FV = P(1 + r \cdot t)$

<u>Simple Interest</u> and <u>Future Value Simple Interest</u>
Use for simple interest, add-on interest loans, and finding credit card finance charge (with average daily balance).
<u>SOLVE FOR:</u> Any variable.

1.2: $FV = P(1 + i)^{nt}$

<u>Future Value Compound Interest</u>
Use for lump sum compound interest investments, inflation (finding price of something in the future or past)
<u>SOLVE FOR:</u> *FV* or *P* only.

1.3: $FV = pmt\dfrac{\left((1+i)^{nt} - 1\right)}{i}$

<u>Future Value Annuity</u>
Use for payments into compound interest account, also used for finding the sum of payments from a COLA payout annuity.
<u>SOLVE FOR:</u> *FV* or *pmt* only.

<u>EXPLORE!</u> Determine when the formulas are appropriate **(without doing any computations)**:

A) When do you use $FV = P(1 + i)^{nt}$ compared to $FV = pmt\dfrac{\left((1+i)^{nt} - 1\right)}{i}$?

B) Which of these formulas are for lump sum investments?

C) Which formula do we use to calculate the interest earned on an annuity investment? Explain.

D) Which formulas are used with compound interest? Explain.

1.4: Dreams Come True – Mortgage/Amortized Loans

Now we've seen compound interest and annuities, and wouldn't it be nice to put those together – like a mathematical dream coming true! For some of you, seeing more complicated formulas can be a problem, but in this case, it actually helps make home ownership possible!

Example 1: Picture this – you have a huge amount of money and you could invest it and make more. But someone you don't know (Steve) wants the money to buy a house of his own. How much money should he pay you back? Well – probably should want to make sure you get as much money at the end of the loan period as you would have had if you just invested the money, right? Let's play this out for a bit. Assume you have $300,000 right now and could invest it at 6% annual interest in the stock market (which goes up or down) or at 4% annual interest fixed. There's benefit to earning the 4% because you won't lose money!

How much money would you have after 30 years if the interest was compounded monthly?
Since this is just a lump sum investment, we would use the compound interest formula.

$$FV = P(1+i)^{nt} \Rightarrow FV = 300{,}000\left(1+\tfrac{0.04}{12}\right)^{12(30)} = 300{,}000\left(1+\tfrac{0.04}{12}\right)^{360} \approx 994{,}049.4044$$

So if Steve wants this money, then he needs to pay you back enough money to have $994,049.41 in order to be fair. So Steve decides to pay you back over time, making monthly payments into his own account that earns 4% annual interest compounded monthly. Because he will make monthly payments, he is investing in an annuity so we can use the annuity formula. We know how much the future value is ($994,049.41) so we can calculate the monthly payment for Steve.

$$FV = pmt\frac{\left((1+i)^{nt}-1\right)}{i} \Rightarrow 994{,}049.41 = pmt\frac{\left(\left(1+\frac{0.04}{12}\right)^{360}-1\right)}{\left(\frac{0.04}{12}\right)} \Rightarrow 994{,}049.41 = pmt(694.0494044) \approx 1432.24589...$$

So Steve needs to make payments of $1,432.25 in order to pay back the money over 30 years.

Does this seem strange? Perhaps giving Steve $300,000 seems strange to you. I mean really, who is this Steve anyway? But bigger than that, it seems that Steve needs to pay back nearly $1,000,000 for borrowing only $300,000. That does seem harsh, right?

But wait! Steve made payments of $1,432.25 for 30 years, or 360 months. This means he actually paid less than $1,000,000: $1,432.25 × 360 = $515,610. This is how much it cost Steve to repay the loan over the course of 30 years. If he made higher payments, he could have paid it off quicker and paid less interest.

Example 2: Steve still wants $300,000 but will pay it back in 15 years. How much does he need to pay each month and how much will he pay back total?

The amount you need back at the end is:

$$FV = 300{,}000\left(1+\tfrac{0.04}{12}\right)^{12(15)} = 300{,}000\left(1+\tfrac{0.04}{12}\right)^{180} \approx \$546{,}090.488211$$

So Steve needs payments of…

$$546{,}090.49 = pmt\frac{\left(\left(1+\frac{0.04}{12}\right)^{180}-1\right)}{\left(\frac{0.04}{12}\right)} \Rightarrow 546{,}090.49 = pmt(246.090488211) \approx 2{,}219.06378 \text{ or } \$2{,}219.07.$$

However, in this case, Steve only has to pay back $2,219.07 × 180 = $399,432.60 on the loan.

Paying it back in 15 years instead of 30 years saves him $515,610 – $399,432.60 = $116,177.40 in interest!

Can we create a formula that ties these concepts together? It must tie together the amount of money the lender would have in the future (Future Value Compound Interest) and the amount of money that the lender would be paying off over time (Future Value Annuity).

$$\text{Amount of Money Lender would have if not loaned} = \text{Amount of money (w/ interest) the borrower repays}$$

$$\text{Future Value (Compound Interest)} = \text{Future Value (Annuity)}$$

$$P(1+i)^{nt} = pmt\frac{\left((1+i)^{nt}-1\right)}{i}$$

For this to work well, we need to have all the information except for one piece. We'll need to find either the *P* or the *pmt* amount. The calculator works really well here to store the calculated portions and then divide as needed. We could use the formula to compute some different monthly payments.

<u>Example:</u> Find the monthly payment on a car loan amount of $20,000 for 5 years at 3.25% annual interest compounded monthly. Then find the total amount paid and the interest paid.

<u>Solution:</u> Plug in the amounts into the formula. Store the left side as "A" and the right side as "B" then divide. $P(1+i)^{nt} = pmt\frac{\left((1+i)^{nt}-1\right)}{i} \Rightarrow \underbrace{20{,}000\left(1+\frac{0.0325}{12}\right)^{60}}_{A} = pmt\underbrace{\frac{\left(\left(1+\frac{0.0325}{12}\right)^{60}-1\right)}{\left(\frac{0.0325}{12}\right)}}_{B} \Rightarrow$

$\underbrace{23523.798668}_{A} = pmt\underbrace{(65.05474464)}_{B} \Rightarrow pmt \approx 361.600046$ Rounding, this payment becomes $361.61.

The payment is $361.61 which means that you would pay $361.61 × 60 = $21,696.60 total for the car. Since the loan was $20,000, the amount of interest would be $21,696.60 – $20,000 = $1,696.60.

<u>EXPLORE!</u> Use the formula to find the monthly payment and the amount of interest paid.

A) A mortgage of $525,000 at 3.75% annual interest compounded monthly for 30 years.

B) A student loan of $42,000 at 4.29% annual interest compounded monthly for 10 years.

We could also use the formula to find the principal loan amount if we know the monthly payment.

<u>Example:</u> If you can afford a monthly payment of $420, determine how much of a loan amount you can afford for 4 years at 3.25% annual interest compounded monthly. Round the loan amount (nearest dollar).

<u>Solution:</u> Plug in the amounts into the formula. Store the left side as "A" and the right side as "B" then divide. $P(1+i)^{nt} = pmt\frac{\left((1+i)^{nt}-1\right)}{i} \Rightarrow \underbrace{P\left(1+\frac{0.0325}{12}\right)^{48}}_{A} = \underbrace{420\frac{\left(\left(1+\frac{0.0325}{12}\right)^{48}-1\right)}{\left(\frac{0.0325}{12}\right)}}_{B} \Rightarrow$

$P\underbrace{(1.138628281)}_{A} = \underbrace{21498.047254}_{B} \Rightarrow P \approx 18880.654569$ Rounding, this loan becomes $18,881.

<u>EXPLORE!</u> Use the formula to find the loan amount.

A) ** You can afford $420 per month for a car, determine how much of a loan amount you can afford for 5 years at 3.25% annual interest compounded monthly. Round loan amount to the nearest dollar.

B) **(L)** You can afford $420 per month for a car, determine how much of a loan amount you can afford for 3 years at 3.25% annual interest compounded monthly. Round loan amount to the nearest dollar.

C) **(R)** You can afford $420 per month for a car, determine how much of a loan amount you can afford for 6 years at 3.25% annual interest compounded monthly. Round loan amount to the nearest dollar.

D) Based on the job you plan to have, you know that you can afford a maximum loan payment of $213 per month to pay back your student loans. What is the maximum amount of student loans you could have at 5.84% annual interest compounded monthly over 10 years (the actual rate for direct unsubsidized loans for graduate students)?

We again could use a table to calculate the payment more quickly as it gives us the factor needed per $1,000 financed through the loan.

AMORTIZATION FACTOR TABLE 3 – PER $1,000 FINANCED

(1% to 8% compounded monthly)

Months	1%	2%	3%	4%	5%	6%	7%	8%
6	167.1531	167.6402	168.1280	168.6165	169.1056	169.5955	170.0859	170.5771
12	83.7854	84.2389	84.6937	85.1499	85.6075	86.0664	86.5267	86.9884
18	55.9964	56.4393	56.8843	57.3314	57.7805	58.2317	58.6850	59.1403
24	42.1021	42.5403	42.9812	43.4249	43.8714	44.3206	44.7726	45.2273
30	33.7656	34.2014	34.6406	35.0833	35.5294	35.9789	36.4319	36.8883
36	28.2081	28.6426	29.0812	29.5240	29.9709	30.4219	30.8771	31.3364
48	21.2615	21.6951	22.1343	22.5791	23.0293	23.4850	23.9462	24.4129
60	17.0937	17.5278	17.9687	18.4165	18.8712	19.3328	19.8012	20.2764
72	14.3155	14.7504	15.1937	15.6452	16.1049	16.5729	17.0490	17.5332
84	12.3312	12.7674	13.2133	13.6688	14.1339	14.6086	15.0927	15.5862
96	10.8432	11.2809	11.7296	12.1893	12.6599	13.1414	13.6337	14.1367
108	9.6860	10.1253	10.5769	11.0410	11.5173	12.0057	12.5063	13.0187
120	8.7604	9.2013	9.6561	10.1245	10.6066	11.1021	11.6108	12.1328
132	8.0032	8.4459	8.9038	9.3767	9.8645	10.3670	10.8841	11.4154
144	7.3723	7.8168	8.2779	8.7553	9.2489	9.7585	10.2838	10.8245
156	6.8386	7.2850	7.7492	8.2312	8.7306	9.2472	9.7807	10.3307
168	6.3812	6.8295	7.2970	7.7835	8.2887	8.8124	9.3540	9.9132
180	5.9849	6.4351	6.9058	7.3969	7.9079	8.4386	8.9883	9.5565
192	5.6383	6.0903	6.5643	7.0600	7.5768	8.1144	8.6721	9.2493
204	5.3325	5.7865	6.2637	6.7639	7.2866	7.8310	8.3966	8.9826
216	5.0607	5.5167	5.9972	6.5020	7.0303	7.5816	8.1550	8.7496
228	4.8176	5.2756	5.7594	6.2687	6.8028	7.3608	7.9419	8.5450
240	4.5989	5.0588	5.5460	6.0598	6.5996	7.1643	7.7530	8.3644
252	4.4011	4.8630	5.3534	5.8718	6.4172	6.9886	7.5847	8.2043
264	4.2214	4.6852	5.1790	5.7018	6.2528	6.8307	7.4342	8.0618
276	4.0573	4.5232	5.0202	5.5475	6.1041	6.6885	7.2992	7.9345
288	3.9070	4.3748	4.8751	5.4069	5.9690	6.5598	7.1776	7.8205
300	3.7687	4.2385	4.7421	5.2784	5.8459	6.4430	7.0678	7.7182
312	3.6412	4.1130	4.6198	5.1605	5.7334	6.3368	6.9684	7.6260
324	3.5231	3.9969	4.5070	5.0521	5.6304	6.2399	6.8781	7.5428
336	3.4135	3.8893	4.4027	4.9521	5.5357	6.1512	6.7961	7.4676
348	3.3115	3.7893	4.3059	4.8597	5.4486	6.0700	6.7213	7.3995
360	3.2164	3.6962	4.2160	4.7742	5.3682	5.9955	6.6530	7.3376
372	3.1274	3.6092	4.1323	4.6947	5.2939	5.9269	6.5906	7.2815
384	3.0441	3.5279	4.0542	4.6208	5.2251	5.8638	6.5334	7.2304
396	2.9658	3.4516	3.9811	4.5520	5.1613	5.8055	6.4810	7.1838
408	2.8922	3.3800	3.9127	4.4878	5.1020	5.7517	6.4328	7.1414
420	2.8229	3.3126	3.8485	4.4277	5.0469	5.7019	6.3886	7.1026
432	2.7574	3.2491	3.7882	4.3716	4.9955	5.6558	6.3478	7.0672
444	2.6955	3.1892	3.7314	4.3189	4.9476	5.6130	6.3103	7.0348
456	2.6369	3.1326	3.6780	4.2695	4.9029	5.5733	6.2757	7.0052
468	2.5813	3.0790	3.6275	4.2231	4.8611	5.5364	6.2438	6.9780
480	2.5286	3.0283	3.5798	4.1794	4.8220	5.5021	6.2143	6.9531

If you were financing $30,000, then you would multiply the appropriate factor by 30.

Example: Find the monthly payment (from the table) for a car loan of $45,000 at 3% annual interest compounded monthly for 4 years.

Solution: First, we need the number of thousands. $45,000 is 45 thousands, so we use "45" for our factor. Next, 4 years is $4 \times 12 = 48$ months. So look up in the table from 48 months over to the 3 % column to find the factor of 22.1343. Now, just multiply: $45 \times 22.1343 = 996.0435$. To complete this, just round up to the nearest penny! So the monthly payment would be $996.05.

EXPLORE! Use the information in the Amortization table to find the monthly payment.

A) ** A home was purchased 10 years ago for $250,000 at 7% annual interest compounded monthly for 30 years. What was the monthly payment? Determine the total amount paid and the interest paid.

B) A home was purchased today for $250,000 at 4% annual interest compounded monthly for 30 years. What was the monthly payment? Determine the amount paid and the amount of interest paid.

C) ** A car is purchased for $30,000 at 3% annual interest compounded monthly for 4 years. What is the monthly payment? Determine the amount paid and the amount of interest paid.

D) **(L)** A car is purchased for $30,000 at 3% annual interest compounded monthly for 6 years. What is the monthly payment? Determine the amount paid and the amount of interest paid.

E) **(R)** A car is purchased for $30,000 at 3% annual interest compounded monthly for 3 years. What is the monthly payment? Determine the amount paid and the amount of interest paid.

Applications

We’ve seen add-on interest loans and now amortized loans. Let’s compare them together and see which is better for us.

Review:

- Add-on interest loans
  - use simple interest on the principal amount
  - compute the interest up front based on rate and time, adds to loan amount
  - divide total by number of months to compute the monthly payment
  - formulas are very easy to use
  - paying more each month shortens time, but not overall payment
- Amortized loans
  - uses simple interest on the remaining balance each month
  - computes interest monthly
  - can be paid off early saving time and money
  - formulas are more complex
  - interest paid is computed by subtracting: “total paid” – “loan amount”.

<u>Example:</u> Take an add-on interest loan for \$27,000 at 2% annual simple interest for 5 years and compare with a \$27,000 at 3% annual interest compounded monthly for 5 years.

<u>Solution:</u>
Add-on interest:

- First, calculate the interest: $I = P \cdot r \cdot t = 27{,}000(0.02)(5) = \$2{,}700$.
- Now find the total by adding this to the loan amount: \$27,000 + \$2,700 = \$29,700.
- 5 years is 60 months, so find the monthly payment: \$29,700 ÷ 60 = \$495.

Amortized Loan:

- Use the formula to calculate the monthly payment:

$$\underbrace{27{,}000\left(1+\tfrac{0.03}{12}\right)^{60}}_{A} = pmt\underbrace{\frac{\left(\left(1+\tfrac{0.03}{12}\right)^{60}-1\right)}{\left(\tfrac{0.03}{12}\right)}}_{B} \Rightarrow \underbrace{31363.65310}_{A} = pmt\underbrace{(64.6467126221)}_{B} \Rightarrow 485.1546$$

- The monthly payment after rounding is \$485.16.
  - You could use the table instead: 27 × 17.9687 = 485.1549, which becomes \$485.16.

Based on this, the payment is lower on the amortized loan even though the interest rate seems higher. This is because the amortized loan calculates the interest each month, so as you pay it off, the balance decreases and the amount of interest goes down. For the add-on interest loan, all of the interest is based on the loan amount for the full 5 years. I would pick the amortized loan!

> **<u>NOTE:</u>** with the Amortized loan, you could make higher payments (like \$500/mo) and it would be paid off sooner and with less total interest. With add-on interest loans, you pay the full interest regardless of the payment amounts. Ouch!

<u>EXPLORE!</u> Compare the two loans listed.

A) ** Compare an add-on interest loan on $150,000 over 10 years at 3.5% annual simple interest with an amortized loan for $150,000 over 10 years at 6.5% annual interest compounded monthly. Analyze the results and see which you would prefer and why.

B) Compare **(L)** an add-on interest loan on $375,000 over 30 years at 4% annual simple interest with an **(R)** amortized loan for $375,000 over 30 years at 5% annual interest compounded monthly. Analyze the results and see which you would prefer and why.

One great application is to see how much you need to pay each month to pay off your credit cards. If you stop making payments, then the credit card becomes an amortized loan! Let's see how much it will cost to

Example: Marianne owes $3,250 on her credit card and she has decided to make no new charges in order to get it paid off. Her interest rate is 18.99%, but she wants to have this paid off in 2 years. How much should she actually pay each month?

Solution: Use the formula for amortized loan and solve for *pmt*.

$$P(1+i)^{nt} = pmt\frac{\left((1+i)^{nt}-1\right)}{i} \Rightarrow \underbrace{3{,}250\left(1+\tfrac{0.1899}{12}\right)^{24}}_{A} = pmt\underbrace{\frac{\left(\left(1+\tfrac{0.1899}{12}\right)^{24}-1\right)}{\left(\tfrac{0.1899}{12}\right)}}_{B} \Rightarrow$$

$$\underbrace{4737.3654159755}_{A} = pmt\underbrace{(28.919487976)}_{B} \Rightarrow pmt \approx 163.8122$$ Rounding, this payment becomes $163.82.

EXPLORE! Determine the amount to be paid each month to pay off the balance.

A) ** If the car loan has a balance of $8,900 and the interest rate is 4.8% annual interest compounded monthly, what is the amount needed to be paid each month to have this paid off in 4 years?

B) **(L)** If the car loan has a balance of $8,900 and the interest rate is 4.8% annual interest compounded monthly, what is the amount needed to be paid each month to have this paid off in 2 years?

C) **(R)** If the home loan has a balance of $238,900 and the interest rate is 3.8% annual interest compounded monthly, what is the amount needed to be paid each month to have this paid off in 12 years?

<u>Formula Rolling Summary</u>

1.1: $I = P \cdot r \cdot t$ and $FV = P(1 + r \cdot t)$

<u>Simple Interest</u> and <u>Future Value Simple Interest</u>
Use for simple interest, add-on interest loans, and finding credit card finance charge (with average daily balance).
<u>SOLVE FOR:</u> Any variable.

1.2: $FV = P(1+i)^{nt}$

<u>Future Value Compound Interest</u>
Use for lump sum compound interest investments, inflation (finding price of something in the future or past)
<u>SOLVE FOR:</u> *FV* or *P* only.

1.3: $FV = pmt\dfrac{\left((1+i)^{nt}-1\right)}{i}$

<u>Future Value Annuity</u>
Use for payments into compound interest account, also used for finding the sum of payments from a COLA payout annuity.
<u>SOLVE FOR:</u> *FV* or *pmt* only.

1.4: $P(1+i)^{nt} = pmt\dfrac{\left((1+i)^{nt}-1\right)}{i}$

<u>Amortized Loan Formula</u>
Use for mortgages, student loans, car loans, credit cards, or other loans that are amortized (where the interest accrues on the unpaid balance, not on the loan amount). Remember the left side is the growth of the lump sum and the right side is the annuity payments. When they are equal, loan is paid off!
<u>SOLVE FOR:</u> *P* or *pmt* only.

<u>EXPLORE!</u> Determine when the formulas are appropriate **(without doing any computations)**:

A) When do you use $P(1+i)^{nt} = pmt\dfrac{\left((1+i)^{nt}-1\right)}{i}$ compared to $FV = pmt\dfrac{\left((1+i)^{nt}-1\right)}{i}$?

B) When do you use $P(1+i)^{nt} = pmt\dfrac{\left((1+i)^{nt}-1\right)}{i}$ compared to $FV = P(1+i)^{nt}$?

C) Which formula do we use to calculate the interest earned on an amortized loan? Explain.

D) Which formulas are used with compound interest? Explain.

1.5: The Trouble with Compound Interest (Mortgages)

One problem with compound interest is that when you're paying it, it seems like a might weight. Indeed, with a home loan, the amount of interest paid each month can be troublesome – the loans are often called simple interest amortized loans, but they seem anything but simple!

Here's why the name is misleading. If you have a mortgage loan for $300,000, we can determine the payment assuming 4% annual interest compounded monthly over a 30 year period.

Review: Determine the monthly payment for the mortgage described above using the table and formula.

Solution: Using the amortized loan formula, we can find the monthly payment to be

$$\underbrace{300{,}000\left(1+\tfrac{0.04}{12}\right)^{360}}_{A} = pmt\underbrace{\frac{\left(\left(1+\tfrac{0.04}{12}\right)^{360}-1\right)}{\left(\tfrac{0.04}{12}\right)}}_{B} \Rightarrow \underbrace{994049.40438}_{A} = pmt\underbrace{(694.04940438)}_{B} \Rightarrow \quad pmt = 1432.24589$$

which rounds to $1,432.25.

Using the table, we would multiply 300 × 4.7742 = $1,432.26.

This payment will guarantee repayment of both the principal amount as well as any interest. For that reason, the monthly payment we calculate is often called the **P & I payment**. With a home loan, you'll still be responsible for taxes and homeowners insurance. We'll assume $350 per month for these other costs, making our total payment $350 + $1,432.26 = $1,782.26. Because interest is calculated on the remaining balance each time, it will decrease with payments. Let's find out how much of that first P & I payment actually goes to paying down the principal with an **Amortization Schedule**, a systematic way of showing how an amortized loan is paid off.

The list below is a comprehensive list of all the information needed for an Amortization Schedule.

Month #	Total Pymt	Taxes/Ins	P & I Pymt	Interest	Principal	Remaining Balance
0						

In order to calculate the interest, we use simple interest $I = P \cdot r \cdot t$. For the first month, we would set up the principal as $300,000 and compute time as one month: $I = 300{,}000 \cdot (0.04) \cdot \frac{1}{12} = \$1{,}000$. The amount remaining after everything else is $432.26.

Month #	Total Pymt	Taxes/Ins	P & I Pymt	Interest	Principal	Remaining Balance
0						$300,000.00
1	$1,782.26	$350.00	$1,432.26	$1,000.00	$432.26	$299,567.74

Then each month, this repeats – the Total Payment stays the same and after each payment, the interest is recalculated. Since the balance went down, then the interest will be slightly lower. But you really should look at this carefully – we paid $1,782.26 and only $432.26 came off the loan. Homes can be expensive!

EXPLORE! Use the computer to create an amortization schedule for the first 10 years of payments on the home. Determine the remaining balance after 10 years of payments as well as the sum of the interest payments.

Now performing calculations like this would be problematic, and mathematicians love to find quicker ways to get the same result. So we always check the remaining balance by thinking again about the big picture.

Amount of Money Lender would have if not loaned = Amount of money (w/ interest) the borrower repays

Future Value (Compound Interest) = Future Value (Annuity)

$$P(1+i)^{nt} = pmt\frac{\left((1+i)^{nt}-1\right)}{i}$$

When we set the two equal to each other, we can find when the loan will be paid off (finding either payment amount or principal amount). But if we know principal and payment, we could use both pieces to find the unpaid balance after tracking the value over fewer months than the total amount. In the 30 year loan above, let's check things out after 10 years.

$$FV = P(1+i)^{nt} = 300{,}000\left(1+\tfrac{.04}{12}\right)^{120} \approx 447{,}249.8047$$

$$FV = pmt\frac{\left((1+i)^{nt}-1\right)}{i} = 1{,}432.25\frac{\left(\left(1+\tfrac{.04}{12}\right)^{120}-1\right)}{\left(\tfrac{.04}{12}\right)} \approx 210{,}898.53281$$

So the bank believes they should have \$447,249.81 and we've only got \$210,898.53 in our account. The difference is the unpaid balance. UB = \$447,249.81 – \$210,898.53 = \$236,351.28. Because of rounding with fractions of pennies, this will be within a few dollars of the amount in the amortization schedule (compare to \$236,349.80).

Unpaid Balance Formula:

$UB \approx P(1+i)^{k} - pmt\dfrac{\left((1+i)^{k}-1\right)}{i}$ where k is the number of payments made so far.

EXPLORE! Practice with the formula and with the amortization schedule.

A) Determine the unpaid balance after 150 months of payments (12.5 years) using the formula and then your amortization schedule.

B) Continue the amortization schedule for the full 360 months. Explain why the loan is not paid off exactly.

C) Adjust the final payment to make sure the loan is paid off exactly. Once done, calculate how much interest you paid total.

D) Using your spreadsheet, create a table that shows the breakdown of your P&I payment. Temporarily hide the columns B, C, and D. Then highlight columns A, E, and F and click on Insert – Stacked Column. When the table appears, click on the lowest region and click delete. That will leave you with only the two parts of principal and interest.

E) How does the amount of interest that you pay change over time? Explain.

F) Now go back and change the total payment on the amortization schedule to an even $2,000. Determine how much interest this will save you and how long until the loan is paid off.

G) Could you use spreadsheets like this to help you plan more with your personal finances? Explain how.

GROUP EXPLORE! Analyze this loan information.

Scott bought a house in Washington state years ago, and at the time, the interest rates were 8.25%. The home had a purchase price of $108,000. He secured the loan for 30 years with additional payment of $228.69. He put 5% as a down payment and paid an additional $2,637.95 in closing costs.

A) **(R)** Determine the loan amount on the home and the monthly P & I payment.

B) **(L)** How much money did Scott have to bring to the closing (counting the down payment and the closing costs)?

C) What is the total monthly payment for Scott's house?

D) Create an amortization schedule spreadsheet and determine how much of the total monthly payment actually went to pay off the loan in the first month? How much total interest would he pay on the home, and how much total (all included) would he have paid for the home?

E) The interest rate listed above was the actual rate in July of 2000. If he had purchased the house with the interest rates in today's market, the rate would be about 4%. Determine the information above if you rework the computation with the new interest rate.
   a. Determine the loan amount on the home and the monthly P & I payment.

   b. How much money did Scott have to bring to the closing (counting the down payment and the closing costs)?

   c. What is the total monthly payment for Scott's house?

   d. Create an amortization schedule spreadsheet and determine how much of the total monthly payment actually went to pay off the loan in the first month? How much total interest would he pay on the home, and how much total (all included) would he have paid for the home?

F) Why is the interest rate so important with home purchases?

<u>GROUP EXPLORE (2)!</u> Continue your analysis.

A) Based on the original loan from the previous page at (8.25%), consult the amortization schedule to determine how long it takes for Scott to pay off 50% of the loan's value.

B) If Scott had an extra $30 each month, add that to the total payment and see how long it takes him to pay it off now. How much interest does he pay if he makes these extra payments? Is this beneficial?

C) After paying the original payment ($999.49 per month) on the loan for 3 years, Scott receives an offer to refinance the home with a lower interest rate. There is a fee of 2% of the remaining balance for this offer, but the interest rate will be 6.5% and he could refinance for 30 years, or he could take 5.75% and refinance for 15 years. Analyze all options to see which is best for Scott (stay with current, refi for 30, refi for 15, refi but make current payment still). Explain below.

<u>Formula Rolling Summary</u>

1.1: $I = P \cdot r \cdot t$ and $FV = P(1 + r \cdot t)$

<u>Simple Interest</u> and <u>Future Value Simple Interest</u>
Use for simple interest, add-on interest loans, and finding credit card finance charge (with average daily balance).
<u>SOLVE FOR:</u> Any variable.

1.2: $FV = P(1+i)^{nt}$

<u>Future Value Compound Interest</u>
Use for lump sum compound interest investments, inflation (finding price of something in the future or past)
<u>SOLVE FOR:</u> *FV* or *P* only.

1.3: $FV = pmt\frac{((1+i)^{nt}-1)}{i}$

<u>Future Value Annuity</u>
Use for payments into compound interest account, also used for finding the sum of payments from a COLA payout annuity.
<u>SOLVE FOR:</u> *FV* or *pmt* only.

1.4: $P(1+i)^{nt} = pmt\frac{((1+i)^{nt}-1)}{i}$

<u>Amortized Loan Formula</u>
Use for mortgages, student loans, car loans, credit cards, or other loans that are amortized (where the interest accrues on the unpaid balance, not on the loan amount). Remember the left side is the growth of the lump sum and the right side is the annuity payments. When they are equal, loan is paid off!
<u>SOLVE FOR:</u> *P* or *pmt* only.

1.5: $UB \approx P(1+i)^{k} - pmt\frac{((1+i)^{k}-1)}{i}$

<u>Unpaid balance on Amortized Loans</u>
Use for mortgages, student loans, car loans, credit cards, or other loans that are amortized (where the interest accrues on the unpaid balance, not on the loan amount). This is the same formula as 3.4 except the exponent is less than the total time.
<u>SOLVE FOR:</u> *UB* only.

<u>EXPLORE!</u> Determine when the formulas are appropriate **(without doing any computations)**:

A) When do you use $P(1+i)^{nt} = pmt\frac{((1+i)^{nt}-1)}{i}$ compared to $UB \approx P(1+i)^{k} - pmt\frac{((1+i)^{k}-1)}{i}$?

B) Why can you not use $P(1+i)^{nt} = pmt\frac{((1+i)^{nt}-1)}{i}$ to compute the unpaid balance?

1.6: The Joys of Compound Interest (Retirement)

As we age, it is typical to consider what to do when we stop working. Retirement! If we've saved up money over time, we'll be able to use that money to live comfortably until retirement ends. Morbid? Yes, a bit. But it's important to think about these things beforehand in order to enjoy retirement.

There are many ways to consider retiring and we can learn from our family. In Scott's family, there were three distinct approaches to retirement.

- Investing in a mattress. Sadly, no investment growth and a small fire created lots of problems.
- Betting on social security. Didn't work so well for grandparents.
- Parents risked on pension and social security. Dad put some money into a retirement account to help with the gap, but not until he was in his late 50's.

Social Security is not a retirement plan – current reports state that by 2033 the Social Security Trust will be broke, meaning that only incoming taxes will be available to cover benefits. So instead of 100% of the benefits, only about 77% of the benefits will be available. We need other choices!

We've seen that we can save lots of money over time with annuities, and now we will consider pulling money out of the account in order to live well. The concept here is similar to something we've seen before: now we have a lot of money and want to trade it for payments over time. It's like a mortgage in reverse!

$$P(1+i)^{nt} = pmt\frac{\left((1+i)^{nt}-1\right)}{i}$$

In this case, we will have *P* as the amount of money in our account, and we can decide how long we need to be paid out.

<u>Example:</u> We have saved up $300,000 and plan to be retired for 20 years. How much will we be paid each month in retirement if we invest at 6% annual interest compounded monthly?

<u>Solution:</u> Plug in the amounts to see about the monthly payments.

$$P(1+i)^{nt} = pmt\frac{\left((1+i)^{nt}-1\right)}{i} \Rightarrow \underbrace{300{,}000\left(1+\tfrac{0.06}{12}\right)^{240}}_{A} = pmt\underbrace{\frac{\left(\left(1+\tfrac{0.06}{12}\right)^{240}-1\right)}{\left(\tfrac{0.06}{12}\right)}}_{B} \Rightarrow$$

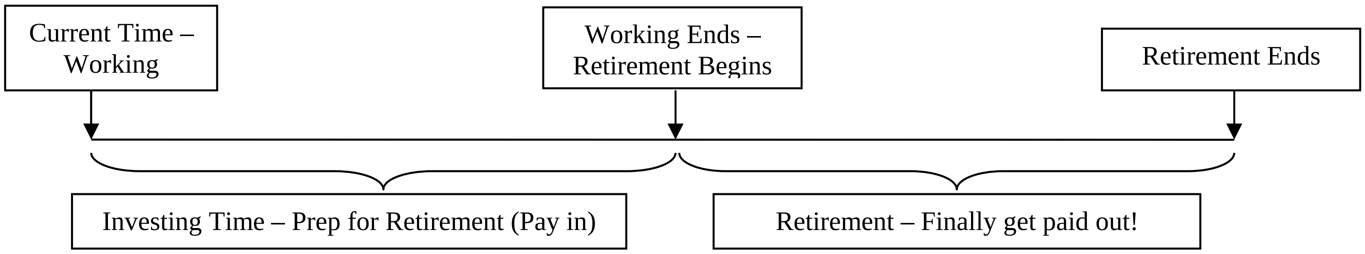
$$\underbrace{993061.342742234}_{A} = pmt\underbrace{(462.0408951614)}_{B} \Rightarrow pmt \approx 2149.2932$$ Rounding, this becomes $2,149.30.

Let's open up the computers and create a spreadsheet to see if this really does work out correctly. The only columns here are: Month Number, Payment, Interest, and Balance. Start with month number 0 and a balance of $300,000. Then for interest, use the previous balance and multiply by 0.06 ÷ 12. When done, fill down to month 240 and you're good. If this is correct, the ending balance should be pretty close to $0.00.

Notice that each month, the balance decreases because we are pulling out more money than the interest we are earning. Because of this, we earn less money each month in interest and eventually will empty the account. If we received $2,149.30 each month and were <u>*not*</u> earning any interest, then we would have to have invested a total of 240 × $2,149.30 = $515,832. Because we are earning compound interest each month, the amount needed is significantly lower. Phew!

It takes a bit to practice this formula – once again, we could use the Amortization Table of Factors to compute monthly payouts more quickly. Checking the table amounts here: 300 × 7.1643 = 2149.29 is pretty darn close!

For the next example involving Margie, a timeline may help. Check out the big points here.



Example: Margie knows that she needs $2,500 per month to live comfortably now. She plans to retire in about 30 years and plans to be retired for about 20 years after that. Determine…

A) how much she needs each month to maintain her current standard of living when she retires 30 years from now (assuming 3% inflation).
B) how much she needs to have in her retirement account in order to be paid out the monthly amount from (A) each month for 20 years if the account earns 5% annual interest compounded monthly.
C) how much she needs to invest each month at 7% annual interest compounded monthly over the next 30 years in order to have the amount in (B).
D) how much interest she earned over the course of the 50 years.

Solution: Each part uses different pieces of what we've learned so far.

A) If she needs $2,500 per month, we need to determine the future amount using compound interest. $FV = P(1+i)^{nt} \Rightarrow FV = 2{,}500(1+0.03)^{30} \approx 6068.156178$ Round this becomes $6,068.16. [This sounds huge, but remember how much gas used to cost per gallon in the 1980's: $0.80/ gallon!]

B) We can use the formula $P(1+i)^{nt} = pmt\frac{\left((1+i)^{nt}-1\right)}{i}$ and solve for P.

$$\underbrace{P\left(1+\tfrac{0.05}{12}\right)^{240}}_{A} = \underbrace{6{,}068.16\frac{\left(\left(1+\frac{0.05}{12}\right)^{240}-1\right)}{\left(\frac{0.05}{12}\right)}}_{B} \Rightarrow P\underbrace{(2.712640285)}_{A} = \underbrace{(2501616.671973)}_{B} \Rightarrow$$

$P \approx 922207.299420$ Rounding, this becomes $922,207.30.

C) For the investment, we'll use the Annuity Formula:

$FV = pmt\frac{\left((1+i)^{nt}-1\right)}{i} \Rightarrow 922{,}207.30 = pmt\frac{\left(\left(1+\frac{0.07}{12}\right)^{360}-1\right)}{\left(\frac{0.07}{12}\right)} \Rightarrow pmt = 755.925594$. Rounding, this becomes $755.93 per month.

D) It helps to picture the situation by taking a step back. Margie invests $755.93 each month for 30 years while she is working. Then, when finished, she is able to pull out $6,068.16 per month to live on for the next 20 years. So in all, she paid in $755.93 × 360 = $272,134.80. With no interest earned, this would be all she has. ☹ Luckily, there was interest and she was paid out $6,068.16 × 240 = $1,456,358.40. The amount of interest she earned over the 50 years is $1,456,358.40 – $272,134.80 = $1,184,223.60.

However, Margie wouldn't know how much to invest each month unless she knew how much she needed at retirement. And she wouldn't know how much she needed at retirement unless she knew how much she needed each month for her standard of living once she retired. And she wouldn't know that unless she found her standard of living and pushed it forward, with inflation, to her retirement date. She had to work backwards! This may seem like a lot of interest, but it really shows the power of compound interest over time which is often referred to as the **time value of money**.

EXPLORE! Determine some information with payout annuities.

A) ** Haley wants to be paid out $5,750 per month when retired. She will retire 25 years from now and plans to be retired for 15 years. Use 4.8% annual interest compounded monthly for both accounts.
   a. How much will she need to have invested at retirement to guarantee 15 years of payouts?

   b. How much will she need to invest each month for the next 25 years of work?

   c. How much interest will she earn over the 40 years total?

B) **(L)** Josue wants to be paid out $3,500 per month over the next 25 years. How much does he need to invest at 4.25% annual interest compounded monthly to make this payout happen?

C) **(R)** Marissa wants to be paid out $3,500 per month over the next 25 years. How much does she need to invest at 5.25% annual interest compounded monthly to make this payout happen?

D) If the interest rate is higher, do we need to invest more or less money to guarantee the payouts? Explain why.

While retirement itself is a pretty big application, another application is lotteries. We learned in unit 2 that investing in lotteries was a pretty bad idea because the expected value virtually guaranteed losing money. However, if there is a payoff, the winnings are paid out as an annuity – scheduled annual payments!

If you win $2,000,000 in a lottery, it may be paid out over 25 payments – one up front, and the next 24 payments paid each year. With 25 payments, each one will be $2,000,000 ÷ 25 = $80,000. Still, not bad, right!

We can determine how much the state needed to purchase this annuity as a cash prize. Let's assume the rate of return is 5% annual interest compounded annually.

$$\underbrace{P\left(1+\tfrac{0.05}{1}\right)^{24}}_{A} = \underbrace{80{,}000\frac{\left(\left(1+\tfrac{0.05}{1}\right)^{24}-1\right)}{\left(\tfrac{0.05}{1}\right)}}_{B} \Rightarrow P\underbrace{(3.2250999437)}_{A} = \underbrace{3560159.9099419}_{B} \Rightarrow P = 1{,}103{,}891.3435478$$

It costs the state $1,103,891.35 to pay for the 24 annual payments, as well as an extra $80,000 for the first payment that doesn't earn any interest. So the grand total is: $1,183,891.35.

If someone was offered the cash prize instead of the annuity, they would be offered $1,183,891.35 instead of the $2,000,000.

EXPLORE! Practice again a few times to find the amount of the cash prize.

A) **(L)** Repeat the calculation for a $2 million lottery, but assuming 4% annual interest compounded annually.

B) **(R)** Repeat the calculation for a $2 million lottery, but assuming 3% annual interest compounded annually.

C) Is the cash prize higher or lower if the interest rate increases? Explain why.

D) Determine the cash prize for a lottery that advertises as $40,000,000 paid in 30 payments (not 25), where the state earns 6% annual interest compounded annually.

EXPLORE! Confirming history:

A) On December 25, 2002, Jack Whittaker won a $314.9 million Powerball lottery and chose the cash option instead of the payments.

   a. **(L)** If there were 25 payments, find his cash value prize if the interest rate was 6%.

   b. **(R)** There were actually 30 payments in his lottery win, and the interest rate was closer to 4.95%. Determine his cash prize now.

B) If Jack had won today, his annuity would be paid over 30 years and the interest rate is running closer to 2.5%. Determine his cash prize if he won the same $314.9 million jackpot today and the payments were all the same.

C) Explain why the cash prize is so much less than the listed annuity prize?

D) If the annuity and cash prizes were close in value, would you suspect high interest rates or low interest rates for the state? Why?

Formula Rolling Summary

1.1: $I = P \cdot r \cdot t$ and $FV = P(1 + r \cdot t)$

Simple Interest and Future Value Simple Interest
Use for simple interest, add-on interest loans, and finding credit card finance charge (with average daily balance).
SOLVE FOR: Any variable.

1.2: $FV = P(1+i)^{nt}$

Future Value Compound Interest
Use for lump sum compound interest investments, inflation (finding price of something in the future or past)
SOLVE FOR: *FV* or *P* only.

1.3: $FV = pmt\frac{\left((1+i)^{nt}-1\right)}{i}$

Future Value Annuity
Use for payments into compound interest account, also used for finding the sum of payments from a COLA payout annuity.
SOLVE FOR: *FV* or *pmt* only.

1.4: $P(1+i)^{nt} = pmt\frac{\left((1+i)^{nt}-1\right)}{i}$

Amortized Loan Formula
Use for mortgages, student loans, car loans, credit cards, or other loans that are amortized (where the interest accrues on the unpaid balance, not on the loan amount). Remember the left side is the growth of the lump sum and the right side is the annuity payments. When they are equal, loan is paid off!

1.6 $P(1+i)^{nt} = pmt\frac{\left((1+i)^{nt}-1\right)}{i}$

Payout Annuities – Fixed payments
This is for payout annuities for retirements, or for lotteries – with fixed payments.
SOLVE FOR: *P* or *pmt* only.

1.5: $UB \approx P(1+i)^{k} - pmt\frac{\left((1+i)^{k}-1\right)}{i}$

Unpaid balance on Amortized Loans
Use for mortgages, student loans, car loans, credit cards, or other loans that are amortized (where the interest accrues on the unpaid balance, not on the loan amount). This is the same formula as 3.4 except the exponent is less than the total time.
SOLVE FOR: *UB* only.

EXPLORE! Determine when the formulas are appropriate **(without doing any computations)**:

A) What's the difference between $P(1+i)^{nt} = pmt\frac{\left((1+i)^{nt}-1\right)}{i}$ used in 1.4 compared to the same formula in 1.6?

B) What formula is used to determine the total interest paid in a payout annuity?

1.7: COLA (not soda)

As we have seen with money, the cost of many items increases over time but $1 is still $1. We may need more money in the future to account for this inflation. As inflation rises, we may need a **cost of living adjustment** to keep up with inflation. This adjustment is called **COLA**, and is usually reported as a percentage.

With COLA, the change is typically an annual adjustment, and it does change the amount of money you'll need. Remember Margie from the previous section? She needed $2,500 to live comfortably in the current year, but over time, the 3% inflation meant that she would need $6,068.16 per month 30 years in the future to have the same buying power. At the time, we calculated how much she needed to have to guarantee payments of $6,068.16 per month each year of her 20 year retirement.

But there is a problem with that. If inflation continues at 3%, then after 10 years of retirement, Margie would need $FV = P(1+i)^{nt} \Rightarrow FV = 6{,}068.16(1+0.03)^{10} \approx 8155.0996$, or about $8,155.10 per month in order to keep the same standard of living. Since our calculations provided her with only $6,068.16 per month, she'd have to take more out of her account and the money would no longer last her 20 years.

Use a spreadsheet to determine how long the money would last if she received a 3% increase to her payments at the end of each year. [Remember that she had an account worth $922,207.30 when she retired.]

So Margie is in trouble, but we can help her out! Since the change with COLA comes each year, we'll just look at her annual payments. $6,068.16 per month would be $6,068.16 × 12 = $72,817.92 per year. We'll use this as our baseline.

EXPLORE! If the COLA is 3% per year, determine Margie's annual and monthly payments for…

A) ** her first year

B) ** her second year

C) **(R)** her 10th year

D) **(L)** her 20th year

The way to compute payout annuities with annual COLA is to compute

$$P = pmt\left(\frac{1-\left(\frac{1+c}{1+r}\right)^t}{r-c}\right)$$

where c is the annual COLA, r is the annual interest rate, and t is the time in years.

Example: Margie really needed an annual payment of $72,817.92 for her first year of retirement. Use this to determine how much money she would need at retirement if the account earns 5% annual interest compounded monthly with an annual COLA of 3%.

Solution:

$$P = pmt\left(\frac{1-\left(\frac{1+c}{1+r}\right)^t}{r-c}\right) \Rightarrow P = 72{,}817.92\left(\frac{1-\left(\frac{1+0.03}{1+0.05}\right)^{20}}{0.05-0.03}\right) = 72{,}817.92(15.96478) \approx \$1{,}162{,}522.33.$$

She needs about $240,000 more at retirement to be able to have the COLA payments.

EXPLORE!

A) $922,207.30 was what we calculated in the previous section. Does it make sense that the new amount with COLA included is higher? Explain why.

B) Now, using the new required amount at retirement, determine what Margie would have to invest each month at 7% annual interest compounded monthly for 30 years to get to this stage. How does this compare with the $755.93 that we previously calculated? Does it make sense that the amounts are different? Explain why or why not.

C) Why is the COLA a better concept than flat payments like we did in the last section?

D) If you were saving for your own retirement, would you want to do flat payments (previous section) or the COLA payments (this section). Explain why.

<u>EXPLORE!</u> With your group, analyze this situation and come up with solutions. Be able to explain your results to the class. <u>NOTE:</u> *this links to Margie's example from section 1.6.*

Leon is looking to save for his future retirement. He is 31 years old now and hopes to retire at age 65, with retirement for about 25 years because his family typically lives to be about 90 years old. Currently, he can live on $3,200 per month comfortably. Inflation is causing prices to increase at 2.5% per year, but he can invest until retirement at 7.2% annual interest compounded monthly, and when he purchases a COLA payout annuity, he can get 5.8% annual interest on it. Determine…

A) how much he will need to make each month to live comfortably at age 65.
B) how much he needs to have at retirement to be able to continue with this standard of living?
C) how much he needs to invest each month starting now until he retires in order to make this happen?
D) how much interest he made over the entire process.

You may use any tools at your disposal including computers/calculators, but no internet for this one.

Even people who are rich need a COLA sometimes. In fact, Powerball and MegaMillions have both changed their payout structures to include a 5% COLA on the annuity winnings each year. Yup - $2,000,000 doesn't buy what it used to, so the COLA allows for each yearly payment to increase a bit which accounts for the higher price of the goods and services. Nice touch!

Here's how the COLA for Powerball works out. Think about any jackpot size – let's pick $90,000,000 because it just happened to be what I looked at when writing this book. If the payout for the annuity is $90 million, then it would be great to know the individual payments. Because the payments are earning annual increases, we can treat this like an annuity:

$$FV = pmt\frac{\left((1+i)^{nt}-1\right)}{i} \Rightarrow 90{,}000{,}000 = pmt\frac{\left(\left(1+\frac{0.05}{1}\right)^{30}-1\right)}{\left(\frac{0.05}{1}\right)} \Rightarrow pmt = 1354629.15722$$

From this calculation, your first payout would be $1,354,629.16. Each year, the payout would increase by 5% so that after 30 payments, a total of $90 million would be paid out.

Confirm this calculation does indeed add up to $90 million using a spreadsheet.

Use this first interest earning payout (5% more than $1,354,629.16, so multiply by 1.05) in the COLA formula with an interest rate of 2.55% (the rate in 2016) and a COLA of 5% to determine the cash value of the $90 million jackpot. [The state listed it at $58.4 million, does it match your calculations?]

EXPLORE! For the initial Powerball jackpot, it starts at $40,000,000. Determine…

A) the first payment if the annuity is chosen.

B) the cash prize if the interest rate is the current 2.55%

C) the amount the winner would get if the cash prize is taken and then charged a tax rate of 39.6% (Federal income tax).

Formula Rolling Summary

1.1: $I = P \cdot r \cdot t$ and $FV = P(1 + r \cdot t)$

Simple Interest and Future Value Simple Interest
Use for simple interest, add-on interest loans, and finding credit card finance charge (with average daily balance).
SOLVE FOR: Any variable.

1.2: $FV = P(1+i)^{nt}$

Future Value Compound Interest
Use for lump sum compound interest investments, inflation (finding price of something in the future or past)
SOLVE FOR: *FV* or *P* only.

1.3: $FV = pmt\frac{\left((1+i)^{nt} - 1\right)}{i}$

Future Value Annuity
Use for payments into compound interest account, also used for finding the sum of payments from a COLA payout annuity.
SOLVE FOR: *FV* or *pmt* only.

1.4: $P(1+i)^{nt} = pmt\frac{\left((1+i)^{nt} - 1\right)}{i}$

Amortized Loan Formula
Use for mortgages, student loans, car loans, credit cards, or other loans that are amortized (where the interest accrues on the unpaid balance, not on the loan amount). Remember the left side is the growth of the lump sum and the right side is the annuity payments. When they are equal, loan is paid off!

1.6 $P(1+i)^{nt} = pmt\frac{\left((1+i)^{nt} - 1\right)}{i}$

Payout Annuities – Fixed payments
This is for payout annuities for retirements, or for lotteries – with fixed payments.
SOLVE FOR: *P* or *pmt* only.

1.5: $UB \approx P(1+i)^{k} - pmt\frac{\left((1+i)^{k} - 1\right)}{i}$

Unpaid balance on Amortized Loans
Use for mortgages, student loans, car loans, credit cards, or other loans that are amortized (where the interest accrues on the unpaid balance, not on the loan amount). This is the same formula as 3.4 except the exponent is less than the total time.
SOLVE FOR: *UB* only.

1.7: $P = pmt\left(\frac{1-\left(\frac{1+c}{1+r}\right)^{t}}{r-c}\right)$

Payout Annuities – Increasing Payments for COLA
This is for payout annuities for retirements, or for lotteries – with increasing COLA payments. This formula uses annual information only.
SOLVE FOR: *P* or *pmt* only.

EXPLORE! Determine when the formulas are appropriate **(without doing any computations)**:

A) What's the difference between the two payout annuity formulas in 1.7 and 1.6?

B) What formula is used to determine the total interest paid in a COLA payout annuity?

1.8: Everyone Loves a Log; You're going to Love a Log!

Solving compound interest problems so far have involved finding the future values. But one tool that we don't have in our tool belt is solving for an exponent in something like: $2{,}097{,}152 = 8^n$.

You see, this isn't like solving $229{,}441 = x^2$ where we can take the square root and solve quickly.

$229{,}441 = x^2 \Rightarrow x = \pm\sqrt{229{,}441} = \pm 479$.

But $2{,}097{,}152 = 8^n$ requires much different tools; indeed, we need to be able to rewrite the equation in a different way that allows us to solve for the exponent. This technique requires a mathematical operator called a **logarithm**. Truly, a logarithm is an exponent.

$a = b^x \Leftrightarrow \log_b a = x$

So the base of *b* becomes the base on the logarithm. Thankfully, our calculators can use this technology to solve for the exponent. There is a log button on the calculator which uses base-ten. Here's how it would work to solve: $10{,}000 = 10^x$.

Example (1): Solve $10{,}000 = 10^x$ for *x*.

Solution (1): $10{,}000 = 10^x \Leftrightarrow \log_{10}(10{,}000) = x$. Because the base is ten, we can just type $\log(10{,}000)$ into the calculator to get the answer of $\log(10{,}000) = 4$. We can check it with $10^4 = 10{,}000$.

If the base is not 10, we can still use the calculator but with a little tweak.

Interactive Example: Solve $2{,}097{,}152 = 8^n$.

This could be rewritten as $\log_8(2{,}097{,}152) = n$, but we don't have a base-8 log button on the calculator. Fortunately, we do have a way to use the calculator with base-10 logarithms.

$$\log_8(2{,}097{,}152) = n \Leftrightarrow n = \frac{\log(2{,}097{,}152)}{\log(8)} \Leftrightarrow n =$$

Check your answer by plugging this into 8^n and see if this does work!

Example (2): Solve $13.7 = 1.05^n$.

Solution (2): Because this is not base 10, we should rewrite and then compute using base 10.

$13.7 = 1.05^n \Leftrightarrow n = \log_{1.05}(13.7) = \dfrac{\log(13.7)}{\log(1.05)} \approx 53.645973064$ which we could round to 53.6460. To check our answer, plug it back into $1.05^{53.6460} \approx 13.7000180$ which shows that it checks out!

EXPLORE! Solve the following equations using your calculator. If the answer is not exact, round to 4 decimal places.

A) ** $11.94^n = 1.097$

B) ** $50(1.05)^n = 270$

C) $500(1.04)^n = 600$

D) $(1.0225)^n = 3.21$

These tools could be used to determine how long it would take to pay off a loan, or how long to save a certain amount of money over time.

Example: If you invest $500 in an account earning 7% annual interest compounded annually, how long will it take you to have $1000?

Solution: Using the formula, we know that it is $500\left(1+\frac{.07}{1}\right)^n = 1{,}000$. First, we can divide by 500 on both sides to rewrite the equation: $500\left(1+\frac{.07}{1}\right)^n = 1{,}000 \Rightarrow \left(1+\frac{.07}{1}\right)^n = 2$. Now use our logarithms to solve for the exponent: $\left(1+\frac{.07}{1}\right)^n = 2 \Leftrightarrow n = \frac{\log(2)}{\log\left(1+\frac{.07}{1}\right)} \approx 10.244768$. So it will be about 10.25 years until we have $1,000.

Check our answer: $500\left(1+\frac{.07}{1}\right)^{10.245} = 1{,}000.01567$. There is a bit off here because of the rounding, but it shows we got the right number!

EXPLORE! Determine how long it would take for the present value to become the future value.

A) ** How long until $30,000 becomes $60,000 at 4% annual interest compounded monthly?

B) ** How long until $50,000 becomes $100,000 at 4% annual interest compounded monthly?

C) How long until $50,000 becomes $200,000 if the rate is 6% annual interest compounded monthly?

D) How long to turn $7,300 into $25,000 if the rate is 4.5% annual interest compounded monthly?

E) How long to turn $7,300 into $25,000 if the rate is 4.5% annual interest compounded annually?

For Love of the Math: *There is a rule of thumb for how long it takes money to double called the rule of 70. Sometimes it is referred to as the rule of 72 when making rough calculations. If the interest rate is between 2% and 14%, you can estimate the amount of time to double by creating 70 ÷ r (in this case, r is a percent but is not turned into a decimal). This is an estimate, but is pretty close!*

Example: How long would it take to double at 4% annual interest? 70 ÷ 4 = 17.5, so it will be about 17.5 years before your money doubles in value. Based on the information in part (A) above, the calculation was 17.35 years… very close to 17.5 years!

If the rate was 10%, then it would take only 70 ÷ 10 = 7 years to double your money!

<u>APPLICATIONS!</u>

Being able to determine how long it will take to pay off an amortized loan, credit card, home loan, student loan, or any major debt, is quite important. However, to this point we only have the ability to calculate a payment for a given amount of time… not the other way around.

There is a bit of algebra involved, but this formula helps us solve for the length of time using logarithms.

$$P = pmt\frac{\left(1-(1+i)^{-nt}\right)}{i} \Rightarrow \frac{P\cdot i}{pmt} = 1-(1+i)^{-nt} \Rightarrow (1+i)^{-k} = 1-\frac{P\cdot i}{pmt} \Rightarrow k = -\frac{\log\left(1-\frac{P\cdot i}{pmt}\right)}{\log(1+i)}$$

<u>Example:</u> Determine how long it will take to pay off an amortized loan of $5,000 making payments of $300 if the interest rate is 16.99% annual interest compounded monthly.

<u>Solution:</u> Plug in the pieces we know, then grab a calculator!

$$k = -\frac{\log\left(1-\frac{P\cdot i}{pmt}\right)}{\log(1+i)} = -\frac{\log\left(1-\frac{5000\cdot\left(\frac{0.1699}{12}\right)}{300}\right)}{\log\left(1+\frac{0.1699}{12}\right)} \approx 19.14434608$$

So it will take a little over 19 months, which rounds up to 20 months. This would be 1 year, 8 months.

<u>EXPLORE!</u> Use this formula to see how long it would take to pay off the given debts.

A) ** How long would it take to pay off a student loan of $44,000 if you can only afford monthly payments of $150 and the interest rate is 4.89% annual interest compounded monthly?

B) **(L)** How long would it take to pay off a mortgage of $484,000 if you can suddenly afford monthly payments of $4,000 and the interest rate is 3.85% annual interest compounded monthly?

C) **(R)** How long would it take to pay off a credit card debt of $6,800 if you can only afford monthly payments of $200 and the interest rate is 14.99% annual interest compounded monthly?

Formula Rolling Summary

1.1: $I = P \cdot r \cdot t$ and $FV = P(1 + r \cdot t)$

Simple Interest and Future Value Simple Interest
Use for simple interest, add-on interest loans, and finding credit card finance charge (with average daily balance).
SOLVE FOR: Any variable.

1.2: $FV = P(1+i)^{nt}$

Future Value Compound Interest
Use for lump sum compound interest investments, inflation (finding price of something in the future or past)
SOLVE FOR: *FV* or *P* only.

1.8: $a = b^x \Leftrightarrow \log_b a = x \Leftrightarrow x = \dfrac{\log(a)}{\log(b)}$

Logarithms
This is used to solve for the exponent in an equation. We can use this to solve for the length of time in a
SOLVE FOR: *x (exponent)* only.

1.8: $k = -\dfrac{\log\left(1 - \dfrac{P \cdot i}{pmt}\right)}{\log(1+i)}$

Length of time to pay off an amortized loan
We would solve this for *k*, which is the number of periods required to pay off an amortized loan. If payments are made monthly, then *k* is months.
SOLVE FOR: *k* only.

EXPLORE! Determine when the formulas are appropriate **(without doing any computations)**:

A) What's the difference between the two formulas in 1.8?

B) How can you solve $FV = P(1+i)^{nt}$ for the length of time?

C) Do we need logarithms to solve any problems involving the formulas in section 1.1? Explain why or why not.

1.9: A Taxing Dilemma

Taxes are a big part of life in America, so it's time to see how the math works with taxes. Our system is a graduated tax system, meaning that the first certain amounts are taxed at one low rate, and as you make more, the new money earned is taxed at gradually higher rates.

It's probably best to get a bird's eye view of how our tax structure works:

1. First, you take all your income. This could come from wages, social security, pensions, retirement accounts, investment properties, etc.
    a. Then you add or subtract based on certain rules. These are called **adjustments**. One example is the ability to take away most costs of moving a long distance for a new job. You won't have to pay taxes on those costs!
    b. Once finished, you have your **Adjusted Gross Income**, or **AGI**.
2. Now, take your AGI and subtract itemized or standard **deduction** and any **exemptions**.
    a. This is where you get credit for the people you supported during the year. The more exemptions, the lower your tax.
    b. Further, you can claim either a standard deduction, or 'itemized' deductions if you have certain items that the government deems worthy of being on a special list. Typically these include gifts/donations to charity, paying interest and taxes on your primary residence, work related costs, etc.
    c. Once finished, you'll have your **taxable income** or **TI**.
3. The taxable income is then used to compute your tax based on your **tax bracket**.
    a. Most people probably use a tax table to look up their tax payment and avoid calculations. If you make more, then you'll have to use the calculations.
4. After the tax is computed, you then calculate any "credits" you may have.
    a. There are items like child care credits, education credits, etc.
5. Credits reduce the tax up front and leave you with a final amount owed – your **remaining tax**.
6. You then determine how much tax was **withheld** from your paychecks.
    a. Compute "withheld" – "remaining tax" and get the final result.
    b. If the subtraction is positive, you get a **refund**!
    c. If the subtraction is negative, then you owe money and have to pay the government!

See – pretty quick! Okay, so there are really a lot of details to determine when computing your taxes, but understanding some important pieces can help you save money.

For college students, there are two places where you can put your educational expenses: Tuition and Fees Deduction (before AGI is computed) or Education Credits (after tax is computed). Which one of these will be better for you? You may have to compute two scenarios and then determine the result.

Some of you who are in college may be asked by your parents if they can "claim" you on their income taxes. Why would someone ask you this?

Graduated tax system (IRS) based on 2015 tax code – **<u>IRS Tax METHOD 1 (Addition)</u>**:

Individual Taxpayers (Single)			
If Taxable Income is Between	The tax due is	Plus this percent	Of the amount over
0 and $9,225	0	10%	0
$9,226 and $37,450	$922.50	15%	$9,225
$37,451 and $90,750	$5,156.25	25%	$37,450
$90,751 and $189,300	$18,481.25	28%	$90,750
$189,301 and $411,500	$46,075.25	33%	$189,300
$411,501 and $413,200	$119,401.25	35%	$411,500
$413,201 and above	$119,996.25	39.6%	$413,200

Married Filing Jointly (MFJ)			
If Taxable Income is Between	The tax due is	Plus this percent	Of the amount over
0 and $18,450	0	10%	0
$18,451 and $74,900	$1,845	15%	$18,450
$74,901 and $151,200	$10,312.50	25%	$74,900
$151,201 and $230,450	$29,387.50	28%	$151,200
$230,451 and $411,500	$51,577.50	33%	$230,450
$411,501 and $464,850	$111,324.00	35%	$411,500
$464,851 and above	$129,996.50	39.6%	$464,850

Head of Household (HoH)			
If Taxable Income is Between	The tax due is	Plus this percent	Of the amount over
0 and $13,150	0	10%	0
$13,151 and $50,200	$1,315.00	15%	$13,150
$50,201 and $129,600	$6,872.50	25%	$50,200
$129,601 and $209,850	$26,772.50	28%	$129,600
$209,851 and $411,500	$49,192.50	33%	$209,850
$411,501 and $439,000	$115,737.00	35%	$411,500
$439,001 and above	$125,362.00	39.6%	$439,000

We will not include Married Filing Separately for space reasons only. Deductions/exemptions are below:

	Single	MFJ	HoH
Standard Deduction	$6,300	$12,600	$9,250
Exemptions	$4,000 per exemption		

<u>Example:</u> Calculate the tax owed by someone who is filing as Single with taxable income of $200,000.

<u>Solution:</u> $200,000 is in the 33% tax bracket, so we will follow the directions there. We will add $46,075.25 to 33% of the amount over $189,300: $200,000 – $189,300 = $10,700.

$46,075.25 + (0.33)($10,700) = $49,606.25. The total tax owed is $49,606 after rounding (nearest dollar).

Even though the $200,000 puts the individual in the 33% tax bracket, the tax is less than 25% of the $200,000 because of the graduated rate.

Graduated tax system (IRS) based on 2015 tax code – **IRS Tax METHOD 2 (Subtraction)**:
NOTE: *This table is used for taxable income over $100,000 – lower amounts are done with a tax table.*

Individual Taxpayers (Single)		
If Taxable Income (TI) is	Take TI Times this percent	Then subtract this
At least $100,000 but not over $189,300	28%	$6,928.75
Over $189,300 but not over $411,500	33%	$16,393.75
Over $411,500 but not over $413,200	35%	$24,623.75
Over $413,200	39.6%	$43,630.95

Married Filing Jointly (MFJ)		
If Taxable Income (TI) is	Take TI Times this percent	Then subtract this
At least $100,000 but not over $151,200	25%	$8,412.50
Over $151,200 but not over $230,450	28%	$12,948.50
Over $230,450 but not over $411,500	33%	$24,471.00
Over $411,500 but not over $464,850	35%	$32,701.00
Over $464,850	39.6%	$54,084.10

Head of Household (HoH)		
If Taxable Income (TI) is	Take TI Times this percent	Then subtract this
At least $100,000 but not over $129,600	25%	$5,677.50
Over $129,600 but not over $209,850	28%	$9,565.50
Over $209,850 but not over $411,500	33%	$20,058.00
Over $411,500 but not over $439,000	35%	$28,288.00
Over $439,000	39.6%	$48,482.00

We will not include Married Filing Separately for space reasons only. Deductions/exemptions are listed below:

	Single	MFJ	HoH
Standard Deduction	$6,300	$12,600	$9,250
Exemptions	$4,000 per exemption		

Example: Calculate the tax owed by someone who is filing as Single with taxable income of $200,000.

Solution: $200,000 is in the 33% tax bracket, so we will follow the directions there. We will multiply $200,000 by 33%, then subtract $16,393.75.

($200,000)(0.33) – $16,393.75 = $49,606.25. The total tax owed is $49,606 after rounding to the nearest dollar.

Both methods get the same result, so we leave it to you as to which one you prefer to use. Some people prefer one over the other! If the amounts are less than $100,000, we'll have to use Method 1. But if they are over $100,000 there is a choice.

EXPLORE (1)! Examine the example below and determine whether which choice is better.

Roberto is filling out his taxes for 2015. He and his wife have income of $67,300 with 2 kids as well. His wife went back to school at CSUSM and they could deal with the education expenses with either a deduction of $4,000 or a tax credit of $2,000. They are also able to take a tax credit of $1,000 per child and will take the standard deduction. The total amount withheld for federal taxes was $6,856.

A) Consider the deduction scenario and run through the tax scenario. Remember that deductions are subtracted prior to AGI. How much tax will they pay with this scenario? Will they get a refund? If so, how much?

B) Consider the tax credit scenario and run through the tax forms. Remember that credits are removed from taxes after the Tax is calculated. How much tax will they pay with this scenario? Will they get a refund? If so, how much?

C) Which option is better for Roberto's family – tax deduction for school, or tax credit?

EXPLORE (2)! Examine the example below and determine whether which choice is better.

Roberto is filling out his taxes for 2015. He has income of $67,300. He also went back to school at CSUSM and could deal with the education expenses using either a deduction of $4,000 or a tax credit of $2,000. He will take the standard deduction. The total amount withheld for federal taxes was $8,982.

A) Consider the deduction scenario and run through the tax scenario. Remember that deductions are subtracted prior to AGI. How much tax will he pay with this scenario? Will he get a refund? If so, how much?

B) Consider the tax credit scenario and run through the tax forms. Remember that credits are removed from taxes after the Tax is calculated. How much tax will he pay with this scenario? Will he get a refund? If so, how much?

C) Which option is better for Roberto – tax deduction for school, or tax credit?

GROUP EXPLORE (3)! Examine the example below and determine whether which choice is better.

Roberto is filling out his taxes for 2015. He has income of $113,000 and is deciding whether or not to get married during 2015. If so, he can claim MFJ status instead of single. His fiancée has no income but does have 1 child (qualifying for the $1,000 child credit). He will take the standard deduction either way. The total amount withheld for federal taxes was $11,263.

A) Consider Roberto filing as single and not getting married in 2015. How much tax will he pay with this scenario? Will he get a refund? If so, how much?

B) Consider Roberto filing as MFJ after getting married in 2015. How much tax will he pay with this scenario? Will he get a refund? If so, how much?

C) Which option is better for Roberto – marriage or not? Explain.

D) Why is this situation sometimes referred to as a **marriage bonus**?

GROUP EXPLORE (4)! Examine the example below and determine whether which choice is better.

Roberto is filling out his taxes for 2015. He has income of $48,000 and is deciding whether or not to get married during 2015. His fiancée, Cynthia, has one child and income of $65,000, filing as HoH (she can claim $1,000 child tax credit). If married, they can claim MFJ status instead of filing separately. They both take the standard deduction either way. The total amount withheld for federal taxes was $5,275 for Roberto and $5,988 for Cynthia.

A) Consider Roberto filing as single and not getting married in 2015. How much tax will he pay with this scenario? Will he get a refund? If so, how much?

B) Consider Cynthia filing as HoH and not getting married in 2015. How much tax will she pay with this scenario? Will she get a refund? If so, how much?

C) Consider Roberto and Cynthia filing as MFJ after getting married in 2015. How much tax will they pay with this scenario? Will they get a refund? If so, how much?

D) Which option is better for Roberto and Cynthia – marriage or not? Explain.

E) Why is this situation sometimes referred to as a **marriage penalty**?

FORMULA SUMMARY:

1.1: $I = P \cdot r \cdot t$ and $FV = P(1 + r \cdot t)$
(use for simple interest or future value simple interest, including add-on interest loans and credit-card average daily balance/finance charge; interest earned on principal only)

1.2: $FV = P(1+i)^{nt}$
(use with compound interest lump sum investments, as well as inflation; interest earned on principle and interest)

1.3: $FV = pmt\dfrac{\left((1+i)^{nt} - 1\right)}{i}$
(investing with payments over time)

1.4: $P(1+i)^{nt} = pmt\dfrac{\left((1+i)^{nt} - 1\right)}{i}$
(amortized loans to find payment or loan amount – home loans, car payments, student loans)

1.5: $UB \approx P(1+i)^{k} - pmt\dfrac{\left((1+i)^{k} - 1\right)}{i}$
(unpaid balance to determine how much the remaining balance is after a certain number of payments)

1.6: $P(1+i)^{nt} = pmt\dfrac{\left((1+i)^{nt} - 1\right)}{i}$
(now used for the payout annuities that are not adjusted for inflation)

1.7: $P = pmt\left(\dfrac{1 - \left(\dfrac{1+c}{1+r}\right)^{t}}{r - c}\right)$
(COLA payout annuities where the payments are annual)

1.8: $a = b^{x} \Leftrightarrow \log_{b} a = x \Leftrightarrow x = \dfrac{\log(a)}{\log(b)}$
(logarithms used to solve for the exponent, which in our case is time! How long to invest a certain amount in order to end with a different amount)

1.8: $k = -\dfrac{\log\left(1 - \dfrac{P \cdot i}{pmt}\right)}{\log(1+i)}$
(This is the way we find out the length of time to pay off a loan with a certain payment like home loans, mortgages, credit cards, etc)

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