Math Fundamentals for Statistics II (Math 95)

Unit 2: Logic and Sets

Scott Fallstrom and Brent Pickett
"The 'How' and 'Whys' Guys"

2nd Edition (Summer 2016)

Table of Contents

2.1: Learning Basic Reasoning with Minesweeper 3
Sometimes the best way to learn a new idea is with a game. In this unit, we've selected two games and each can be played on a cell phone or computer. The ideas of logic start to appear as we must justify our moves and why we make them.

2.2: Fallacies in Reasoning/Argument 7
Seeing many arguments outside the math classroom shows us that they not only exist, but that we must be wary of the reasoning used. Politics, commercials, and other areas are show to use different types of false reasoning to convince us to vote a certain way or buy a certain product.

2.3: Types of reasoning 14
In other courses like biology, physics, and chemistry, one type of reasoning is used. In mathematics, a completely different type is used because of additional information. We show the limitations of different types of reasoning.

2.4: Formal Logic – Statements and Quantifiers. 17
Being able to make comments about general ideas requires breaking it into pieces. Is something true all the time, some of the time, or none of the time? We explore this with the new idea of negating an idea.

2.5: Formal Logic – Operators (Truth Tables). 21
Formal logic is a bit more precise and the notation must come along accordingly. Here we describe compound or binary operators, which are the operators that require two statements. We explore all possibilities by organizing our thoughts into a list called a truth table.

2.6: Expanded Conditionals and Bi-Conditionals. 27
Conditionals and bi-conditional statements are used often and misunderstood even more frequently. This section provides a more in-depth look at those statements and how to use them correctly.

2.7: Basic Arguments – Using Logic 33
With the idea of an argument and the fallacies from previous sections, we now begin to determine whether arguments are valid or not using logic.

2.8: Arguments – Using Venn Diagrams 37
With the idea of an argument and the fallacies from previous sections, we now begin to determine whether arguments are valid or not using pictures called Venn diagrams to organize our thoughts.

2.9: Learning Deductive Reasoning with Mastermind 44
Our second game is covered here and the deductive reasoning covered in previous sections is used to solve problems in a game-like situation.

2.10: Sets – The Basis for Counting 48
While it may seem like a repeat of previous sections, this section talks about grouping objects specifically. Simple idea but the techniques grow in complexity. We also cover new notation for counting objects in a set.

2.11: Sets – Basic Operators and Venn Diagrams 53
The idea of sets is fairly basic, and now we get to see about operators. The connection between the new operators and the operators used in the logic section is explicitly stated.

2.12: Using Sets to Solve Problems 61
Sets are excellent tools to solve problems that often come up in statistics courses. Some sample problems are covered.

2.13: Basic Counting Techniques 64
Now that we can group objects in sets, we can describe ways to count the objects as well as ways to expand these to new problems. A calculator is absolutely essential here!

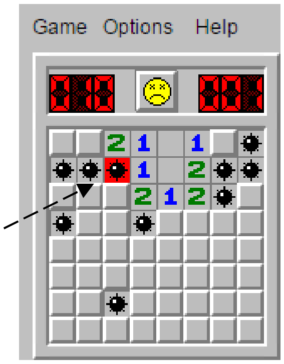
2.14: Advanced Counting Techniques 69
Counting techniques expands to illustrate some great ways that will help us in coming units. The idea of a combination is extremely helpful and again, a calculator makes this manageable!

2.15: Summary 73

INDEX (in alphabetical order): 75

2.1: Learning Basic Reasoning with Minesweeper

Game #1: **Minesweeper**. Go to http://www.freeminesweeper.org/
Or Google Play: choose "Minesweeper Classic" by Juraj Kusnier



Rules/description:

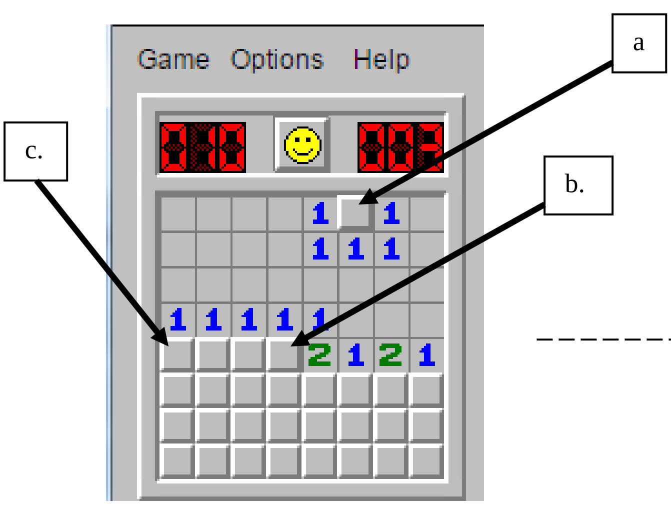
1. Boxes fill the screen. A box hides what is under it and the options are:
    a. Nothing (blank).
    b. A number (indicating how many bombs that particular box is touching)
    c. A bomb (which if you open the box will lose the game).
2. There is a timer in the upper right, and a counter on the number of bombs unmarked in the top left.
3. You mark something as a bomb using the right mouse button and open a box by clicking the left mouse button.
4. So when you start a game, you open any box to begin.
    a. If you click a number, then only that box opens.
    b. If you click a blank space, then all the area opens showing many numbers.
5. If you click a bomb, click on the face in the middle to start again! You can adjust between Beginner (10 bombs in 8-by-8 grid), Intermediate (40 bombs in 16-by-16 grid) and Expert (99 bombs).

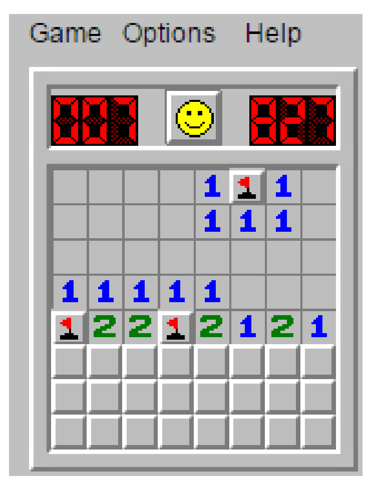
Example: Solve the puzzle that is started here:

The solution will be in parts:
Solution (part 1):

a. The top box must be a bomb, since a box marked with a "1" is touching one bomb.
b. This must be a bomb because a 1 is touching only one bomb. So the boxes next to it can't be bombs. (circles)
c. Because the two nearby are not bombs, this must be a bomb.

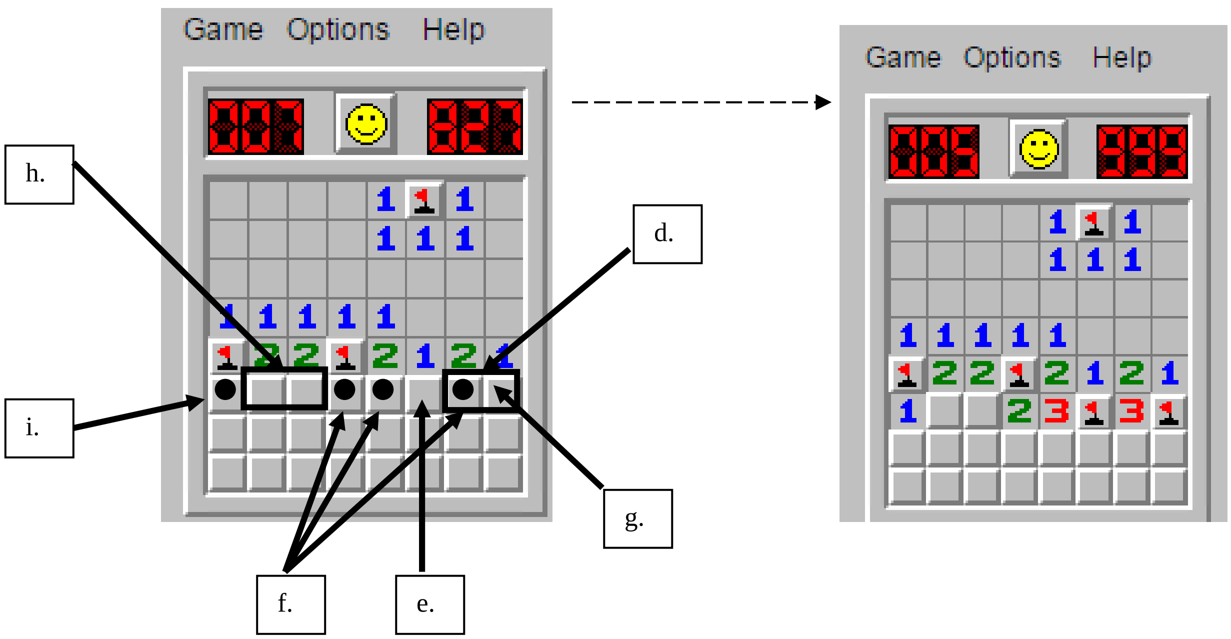
We will mark these and show the result.





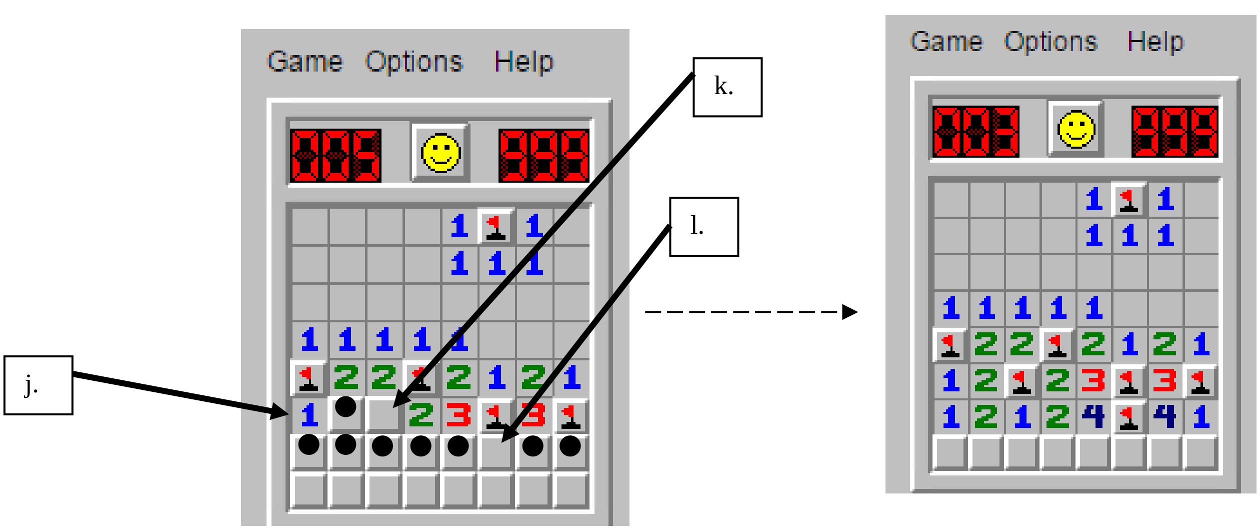
Solution (part 2):

d. Only one of these can be a bomb because of the 1 against the edge, but because the 2 must touch two bombs…
e. This must be a bomb.
f. …which means that all of these are not bombs…
g. Making this one a bomb for sure.
h. And since one of these two must be a bomb (but we're not sure which one)
i. Then this must not be a bomb.



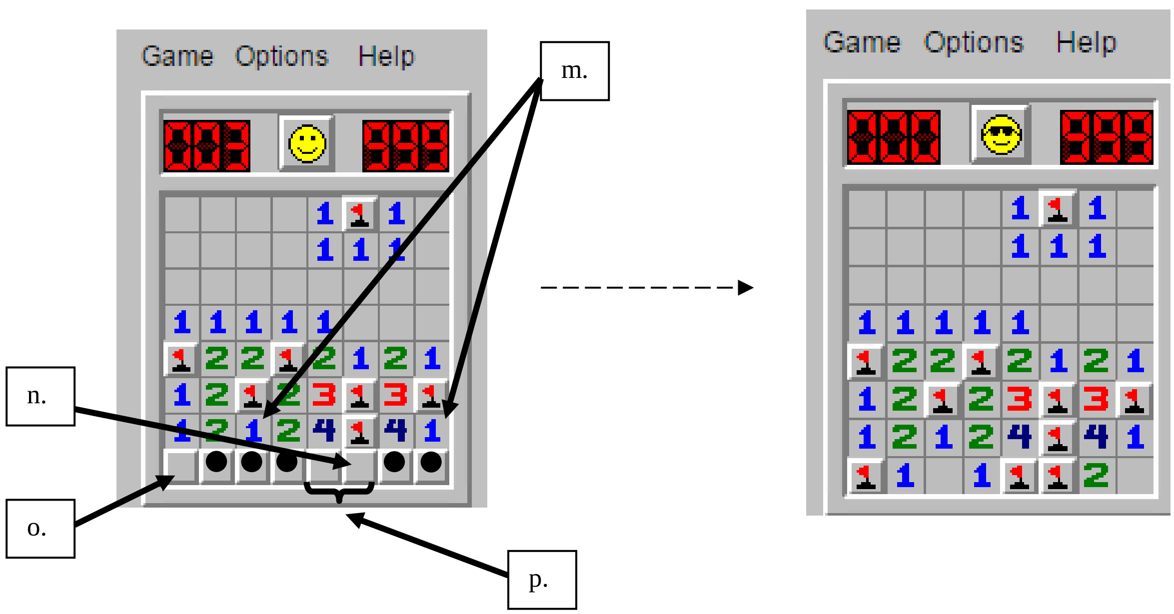
Solution (part 3):

j. Because of the 1 already touching a bomb, none of the other boxes it touches can be bombs.
k. …which means this must be a bomb, and the others below the 2 can't be bombs.
l. Because of the 3 that is touching two other bombs, this one must be a bomb, and because of the other 3, the others must not be bombs.



Solution (part 4):

m. Because of the 1 already touching a bomb, none of the other boxes it touches can be bombs.
n. Because of the 4 on the right touching three bombs only, this must be a bomb.
o. Since the 1 that is in the bottom left is not touching a bomb yet, this must be a bomb.
p. Lastly, there is a 4 touching only two bombs with only two boxes left, so both must be bombs.



EXPLORE (1)! Using either the computer in the desk, or your cell phone, open up minesweeper.

A) Create a custom game that is 10 × 10 with 6 mines. Play this a few times to get the hang of it. When you feel comfortable move on.
B) Create a custom game that is 10 × 10 with 7 mines. Play this a few times to get the hang of it. When you feel comfortable move on.
C) Create a custom game that is 10 × 10 with 8 mines. Play this a few times to get the hang of it. When you feel comfortable move on.
D) Create a custom game that is 10 × 10 with 9 mines. Play this a few times to get the hang of it. When you feel comfortable move on.
E) You're ready to play Beginner. Play this a few times to get the hang of it. When you feel comfortable move on.
F) If you're feeling good, try a few that are intermediate as well. Intermediate is 16 × 16 with 40 mines. You can always try a custom version with around 20 mines to start getting the hang of the larger puzzle.

If you are stuck, ask someone near you or call over the instructor.

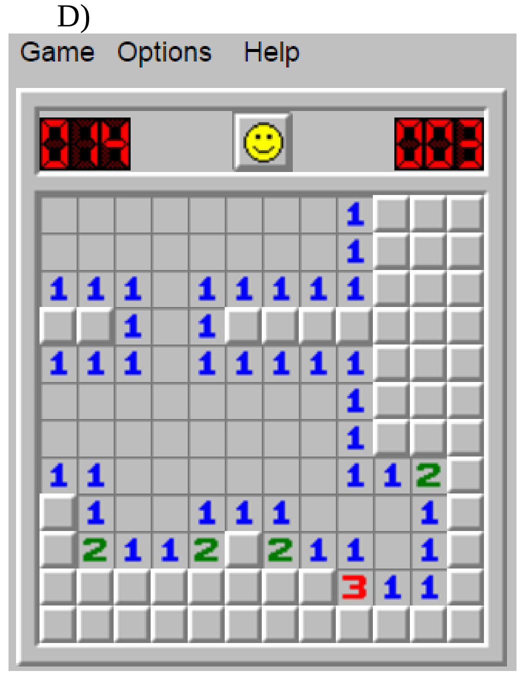
<u>EXPLORE (2)!</u> Based on the following games of Minesweeper, find all the squares that are bombs. Do you have enough information to solve the puzzle? If there are any squares that you're not sure about, can you explain why. Discuss your results with your neighbor. Write a "B" on any bombs, an "X" on any that are not bombs, and leave the ones you are not sure about open.

A) **

B) **(L)**

C) **(R)**

D)



2.2: Fallacies in Reasoning/Argument

As we move into our second course, the ability to think and reason clearly becomes more important and is a bigger focus. Most of this comes from our ability to use numbers (number sense) and to become citizens who can use numbers to present and justify a particular position.

Argumentation often occurs in life, both personal and professional, and before we start talking about the logic and reasoning skills needed to understand good arguments, we look at the fallacies in reasoning which can be deceptive arguments. As we study **logic**, we begin to study the ideas, concepts, and principles to separate good reasoning (correct) from bad reasoning (incorrect). While we mainly start with the good, this time we start with the bad and the ugly – fallacies.

An **argument** requires a number of **premises** (facts or assumptions) which are followed by a **conclusion** (point of the argument – what you are claiming is true). The premises are used as justification for a conclusion. A conclusion which is correctly supported by the premises is known as a **valid argument**, while a **fallacy** is a deceptive argument can sound good but is not well supported by the premises.

NOTE: Since we are starting with structure in general, we won't be looking at the truth of the statements or premises at this point. Later we'll look to see if the premises are actually true, but for now, we won't get caught up in that discussion.

For arguments in this class, we are really discussing ways to convince someone else that our idea is true. How can we convince someone? We should have some items we believe prove our point – statements, pictures, facts, etc. Sometimes, the way an argument proceeds brings to light some flaws in the reasoning or the methods used. We'll discuss two types of fallacies in this course, and then give a generous list of the different types.

The two types depend on what the fallacy is related to. In some cases, the argument relates back to premises that have nothing to do with the conclusion being made, while the other type is based on false premises. Let's go into a bit more detail:

- Premise is not related to the conclusion
    - Called fallacies related to relevance.
    - Used a lot in commercials to convince you to purchase something.
    - An example would be Colonial Penn Life Insurance. They hired Alex Trebek, the famous host of Jeopardy, to describe their product on TV. He spoke to seniors about a rate-lock guarantee where their life insurance was safe over time and the rates would never increase.
    - While their product may have met all of those claims, having Alex Trebek describing it is a way to convince people that it is a good plan. He's not a life insurance expert, so why should anyone trust him about life insurance?
- Think of an example of this type – do you have any experience with these types of arguments (or commercials)?

- Premises (assumptions) are false
    - Called fallacies related to assumptions
    - Used a lot in politics
    - An example would be Eric Cartman (South Park) wanting to bulk up and be more muscular on TV. So he watched a commercial where a very muscular man told him that he should take "Weight Gain 4000" to bulk up: *{Hey!! You need to get in shape fast?! Wanna look your best?! Tired of the other guys getting all the chicks?! Are you tired of being a 90 pound weakling?! Then bulk up quick with Weight Gain 4000!! With over 4,000 grams of saturated fat per serving, its patented formula is designed to enter the mouth, go directly to the stomach where it is distributed to the bloodstream. Now available in stores everywhere. Get some today and say with me: 'BEEFCAKE!'}* Cartman did take the product, but bulked up in a very different way – he added about 100 pounds of fat to his body!
    - You see that a man who worked out a lot and took the product was very muscular. Eric assumed this to mean that if he took the product, he too would be muscular. His false assumption created a severe weight problem!
- Think of an example of this type – do you have any experience with these types of arguments (or commercials)?

CAUTION: We will now discuss many types of fallacies under these two headings. *Do not read these to memorize.* Any question put on a test involved with these types of fallacies will be on the take home and you will be able to refer to this book as a guide. The goal of presenting them is so you are more knowledgeable in future arguments as well as more aware of these types of fallacies in your daily lives. Sometimes assumptions are not clearly stated – we must be careful what assumptions we make in reading an argument too. Try to not put your own thoughts onto the argument – just use what it says!

FALLACIES RELATED TO RELEVANCE – the premises are not relevant to the conclusion.

1. Appeal to Ignorance: This approach uses a lack of proof. You may think of this one as "A lack of evidence doesn't mean it doesn't exist." Here are some examples:
    a. There has not been one report of UFOs shown to be true. Therefore, UFOs don't exist.
    b. Nothing can prove that UFOs don't exist, so we must accept that at least some of the reports about UFOs are true.
    c. Since the class has no questions about section 1, the students must completely understand it.
2. Appeal to Emotion/Fear: Making no appeal to logic at all, but tying in to love, happiness, or other positive emotions (or negative emotions). Here are some examples:
    a. Choosy moms choose Jif.
    b. If you elect Clinton, we will be overrun by immigrants and the economy will collapse.
    c. Hallmark cards; when you care enough to send the very best.
3. False Authority: Someone being credible in one area discussing another area. Here are some examples:
    a. Joe Namath told me that the best pantyhose to buy is Beautymist.
    b. Roger Federer says that Gillette makes the best razors.
    c. I'm Pat Boone, famous recording star. And the world economy is about to collapse so you need to buy gold, like I am.

4. Straw Man: This fallacy focuses on distorting someone's views to convince you to oppose them. A straw man (scarecrow) is a pretty bad representation of a real person, just like the straw man argument is a pretty bad representation of the person's actual beliefs. Here are some examples:
   a. Some Democrat senators have recommended cutting back on defense spending. If we have no military, then we will be defenseless and our country will be overrun.
   b. The governor recommended releasing low level prisoners to save money – prisoners convicted of minor drug offenses. You should vote for me as I do not reward drug users.
   c. Bernie Sanders is pushing to have tax dollars used to help poor people. I guess he doesn't value hard work and prefers handouts.
5. Appeal to popularity (common practice): Our position is popular and therefore it is correct. Here are some examples:
   a. People buy more McDonald's French Fries than any other restaurant. Therefore, McDonald's has the world's best fries.
   b. You can break the speed limit law on the freeway because everyone does it.
   c. Apple makes the best cell phone because more i-Phones are sold than any other type.
6. Personal Attack: This fallacy is often brought up as a counter to an argument. An example is when a person brings forward a proposal and then is personally attacked for something other than the argument. Here are some examples:
   a. My opponent claims to have a way to balance the budget and not cut services. But how can we trust him because he has cheated on his wife?
   b. I'm not an alcoholic because you drank a lot last weekend too.
   c. Lisa, I know that you think I'm a hoarder because my house has been condemned and there is 8 tons of garbage in the house. But you have so many cats and a dirty house all the time.
7. Diversions (Red Herring): This type of fallacy is designed to distract from the claims being made. If you've seen the movie "UP" it is the type of distraction that the dog often has… "Squirrel!" Here are some examples:
   a. We can't continue research with stem cells because of the ethics involved. We can't afford to have unethical decisions made in this area because the family and solid ethical decisions are the key to the strength of our country.
   b. Yes, Julia, mass shootings are troublesome. However, freedom is a key part of this country and we need to preserve our freedoms.
   c. Dad, have you been thinking about your retirement accounts – maybe we should move some money into a less risky portfolio. *Son, I thought about that and I know that mustard is the best thing to put on a cheese sandwich. And Von's had mustard at 2 for $1 last week?*

Now that you have an idea of many of the types, we're going to have you practice identifying types of fallacies. It's important to read each carefully, determine the conclusion, and then find which type fits.

EXPLORE (1)! Determine which type of fallacy of relevance is being used in each case.

A) ** Marvin thinks that Ken Griffey, Jr. doesn't belong in the Hall of Fame. He was arrested for shoplifting when he was a kid so I wouldn't trust anything he says.

   See how this is claiming that Ken Griffey, Jr. should not be in the Hall of Fame. But why? What are the reasons for this conclusion?

B) **(L)** The best cheese is Tillamook cheese, and I know this because of my years playing college lacrosse.

Conclusion:

Type of fallacy:

C) **(L)** The first time you hug your baby, the rest of the world will slowly melt away and your life will be forever changed. At Huggies, we understand there is nothing more powerful than that moment and every moment that follows. That's why our diapers are the best.

Conclusion:

Type of fallacy:

D) **(L)** Ford trucks are better than Chevy because Ford is the best-selling truck for the last 5 years.

Conclusion:

Type of fallacy:

E) **(R)** Lance Armstrong: "I do not use drugs. Want to know how I can show you that, here's how – every single drug test I've taken has been clean. So yeah, I'm clean."

Conclusion:

Type of fallacy:

F) **(R)** Yes, we should discuss the money that is missing from accounting. But I think it is really important to see that there is a huge rat problem here. Maybe we can get Milton to move his desk to the basement and work on the rat problem.

Conclusion:

Type of fallacy:

G) **(R)** Barbara voted to repeal the estate tax on the wealthy. So she must really hate poor people and want to give your money to the wealthiest Americans.

Conclusion:

Type of fallacy:

FALLACIES RELATED TO ASSUMPTIONS – false premises are assumed, conclusion based on these.

1. False Cause: Two things happen, but the idea that the first occurred is used to claim evidence that it caused the second. Here are some examples:
    a. I saw a 3-d movie about racing cars and my headache went away. 3-d movies cure pain.
    b. Two bodybuilders that I know consume about 5,000 calories per day. If I start eating 5,000 calories per day I will be in great shape.
    c. I wear shorts every day and it has not snowed since I started doing this. I must need to keep doing it to ward off the snow.
2. Circular Reasoning (begging the question): In this type, an argument basically restates the premises as the conclusion. Here are some examples:
    a. Drugs are illegal because they're bad for people. Drugs are bad for people since they are illegal.
    b. Creating free education for everyone is a bad idea because it is wrong to have free education.
    c. You need to stay with me because people who are in love stay together, and we are still together because we're in love.
3. Hasty Generalization: This fallacy has a conclusion that is based on a few examples that may not be typical. Here are some examples:
    a. My math teacher at Palomar was great last term, and this term my math teacher at MiraCosta is great too. Therefore, all math teachers in California are great.
    b. An article from 3 teachers said that "Common Core" math is bad. We need to get rid of "Common Core" math.
    c. In Minesweeper, the past 4 games had a bomb to the left of a "3." I'll mark the space left of the "3" in this game as a bomb too.
    d. The fail rates in class A are high. This means the students are not provided enough support.
4. False Dilemma (limited choice): This fallacy focuses on only having a few options. Many times the type of question can lead to the limited choice. Here are some examples:
    a. Yes or no, have you stopped beating your kids?
    b. You need to buy this $2,000 computer or you will fail your classes.
    c. If you don't buy this toothpaste then you will have tooth decay.
5. Middle Ground: If there are two extreme positions, then a middle one is best. Here are examples:
    a. Bernie Sanders is in favor of no guns but Donald Trump wants to arm everyone. The best policy is to have some guns.
    b. Senator Rankin wants to raise taxes by 10% and Senator Williams wants to raise taxes by 4%. Raising them by 6.5% would be the best option.
    c. Some countries have a legal drinking age of 21, while some have it as 16. The best age would be 18 years old.
6. Slippery Slope: If something happened on a small level, before long it will happen on a larger scale. Here are some examples:
    a. If we legalize marijuana, then all drugs will be legal soon and the murder rate will go up.
    b. If we keep raising taxes on the rich, then soon all the wealthy people will move to a different country and our country will be bankrupt.
    c. If you don't care about our curfew policy, then you'll end up not caring about any of the rules in society and you'll be homeless and pregnant before you know it.

7. <u>Division fallacy:</u> A group has some property, so therefore any individual item in that group must have that property. Here are some examples:
    a. The highest test scores in math occur in Singapore, China, Japan, and Korea. So the Asian student in my math class is going to score higher than anyone else.
    b. America uses more gas per person than any other country. So Peter, who is American, will use more gas than Julie, who is Swedish.
    c. Teachers make less than Engineers based on the recent report. Lucy, a teacher, must make less than Karl, an engineer.
8. <u>Gambler's fallacy:</u> Something is happening more than it should or at a higher rate than normal, so it must end soon. Here are some examples:
    a. Hasn't rained in 4 weeks, we are due for a good shower.
    b. My car is running so well right now, it is going to break down any day.
    c. I bought $40 worth of tickets for the Powerball when it was $948 million. I know that I'm due for a win, so I will spend $100 on tickets for the next drawing.

<u>EXPLORE (2)!</u> Determine which type of fallacy of assumption is being used in each case.

A) ** My income tax is too high because I have to pay so much in income tax.

<u>Conclusion:</u>

<u>Type of fallacy:</u>

B) **(L)** The state has cut taxes over the past few years, and if this keeps up, there won't be any money to fix roads and the infrastructure will collapse.

<u>Conclusion:</u>

<u>Type of fallacy:</u>

C) **(L)** Grandpa lived until he was 95, and he ate bacon every day and smoked cigars. So I know that I'll be okay eating bacon and smoking cigarettes.

<u>Conclusion:</u>

<u>Type of fallacy:</u>

D) **(L)** My parents think that I deserve a 100% on this paper, but you only scored me as a 60%. I think that changing the score to 80% would be the best way to resolve this.

<u>Conclusion:</u>

<u>Type of fallacy:</u>

E) **(L)** People who live in European countries on average know 3 or more languages. Pierre, who grew up in France, must know Spanish.

<u>*Conclusion:*</u>

<u>*Type of fallacy:*</u>

F) **(R)** The last 5 dates I found on the Plenty-of-Fish app were just trying to hook-up. I'm due for someone who is husband material, so I'm going to make 3 dates for this week.

<u>*Conclusion:*</u>

<u>*Type of fallacy:*</u>

G) **(R)** Every guy I ever got in a car with was a horrible driver. So men are worse drivers than women.

<u>*Conclusion:*</u>

<u>*Type of fallacy:*</u>

H) **(R)** The ex-New York Governor exercised his 5th Amendment right to not testify. He must be guilty.

<u>*Conclusion:*</u>

<u>*Type of fallacy:*</u>

I) **(R)** Every homeless man I've seen has really long hair, and they never use shampoo. Since I'm balding, I will quit using shampoo and that way I'll have long hair again.

<u>*Conclusion:*</u>

<u>*Type of fallacy:*</u>

<u>**For Love of the Math:**</u> *As example of how a fallacy can cripple an argument, we'll demonstrate one from the Supreme Court:* Whole Woman's Health v. Hellerstedt. *In this case, the state of Texas had decided to require massive changes to certain health clinics – a decision that essentially shut down a vast number of the clinics. The state claimed that these changes were absolutely critical to protect women's health, and that it was not an 'undue burden' because women in El Paso could just cross the state line and use a clinic in New Mexico. Supreme Court Justice Ruth Ginsburg asked a few questions: "It's odd that you point to the New Mexico facility, so if your argument is right, then New Mexico is not an available way out for Texas, because Texas says: to protect our women, we need these things. But send them off to New Mexico (to clinics with more lenient standards) and that's perfectly alright. Well, if that's alright for the women in the El Paso area, why isn't it right for the rest of the women in Texas?" Indeed, relying on the existence of clinics with lower standards to justify the constitutionality of a Texas law requiring massive changes, just didn't work.*

A careful study of logic is critical for attorneys and judges alike, but also very helpful for regular folks too!

2.3: Types of reasoning

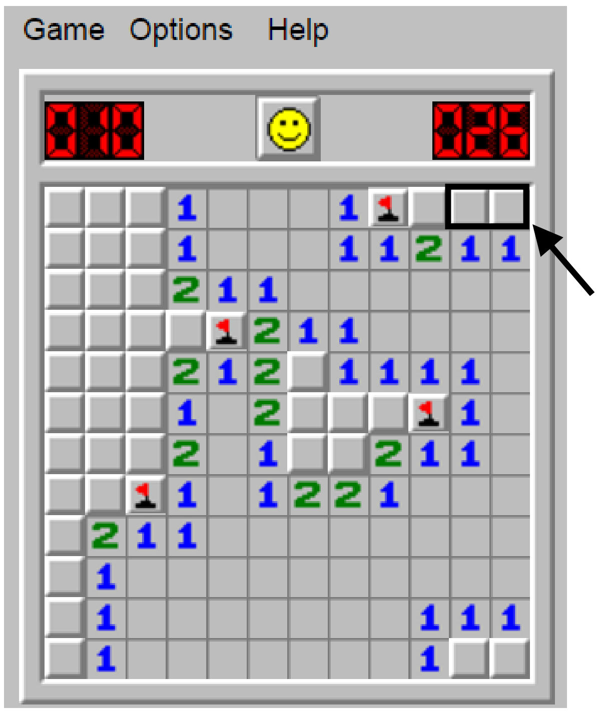
Based on some of the fallacies, we can see that solid reasoning requires good thinking skills. For what we've done in Math 52, there were two main types of reasoning: inductive and deductive.

The process of seeing some specific information and expanding that to make a general conclusion is known as **inductive reasoning**. This was what we used when asked to find the next term in the sequence of numbers: 5, 7, 9, 11, … Because we weren't sure what it was, we had to pick something and figure it out. With inductive reasoning, the conclusion is not guaranteed to be true because there could be many possibilities! Here are some possibilities:

- 5, 7, 9, 11, 13, 15… (if we are adding 2 to the previous term)
- 5, 7, 9, 11, 9, 7, 5, … (if the sequence goes up then down)
- 5, 7, 9, 11, 1, 3, 5, … (if the sequence is adding 2, but based on a 12-hour clock)
- 5, 7, 9, 11, 5, 7, 9, 11, … (if the sequence repeats)

However, if you were asked to find the next term in the *arithmetic* sequence 5, 7, 9, 11, … then there would be only one solution. Because this method uses a general idea (arithmetic sequences) and applies it to a specific instance, we know the resulting conclusion is guaranteed. This process is known as **deductive reasoning**. *NOTE: you must have a general statement that is true in order to use deductive reasoning.*

Here's an example with Minesweeper to illustrate the differences.



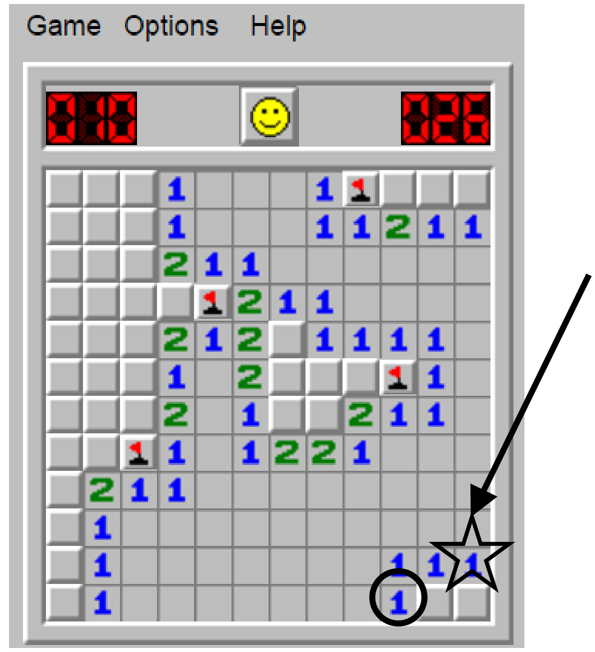
Damien says that in this game, he can see that everyplace there is a 1 with a box over it, that box is a bomb. So he marks the both of the two outlined boxes as bombs.

A) Would you say that his conclusion is correct in this case?

B) Could you think of a time when he might be correct?

C) Will this type of thinking guarantee the correct result?

Now consider Jesenia, who knows that any number touches exactly that many bombs. So a "1" will touch exactly one bomb. Jesenia sees that in this game, there are two unopened boxes in the lower right corner. The circled 1 is touching only one box, so that box must be a bomb. The starred number must also touch only one bomb, so the box below it can't be a bomb. So she marks the left box as a bomb and clicks to open the other outlined box.



D) Would you say that her conclusion is correct in this case?

E) Could you think of a time when she might not be correct?

F) Will this type of thinking guarantee the correct result?

Minesweeper wrap up: Which type of reasoning is used by Damien and by Jesenia?

Interactive Examples: Determine whether the example demonstrates deductive (***D***) or inductive (***I***) reasoning. Circle your choice.

A) ***D*** ***I*** Jolene is given a list of numbers: 3, 5, 7, … She is asked to find the next number in the sequence and says the next number is 9 because the sequence goes up by 2 each time.

B) ***D*** ***I*** Marcus is given a list of numbers: 3, 5, 7, … He is asked to find the next number in the sequence and says the next number is 11 because the sequence lists the odd prime numbers.

C) ***D*** ***I*** David is given a list of numbers: 3, 5, 7, … He is asked to find the next number in the arithmetic sequence and says the next number is 9 because the sequence goes up by 2 each time.

D) ***D*** ***I*** Martha watched Star Wars Episode 1 and found it to be terrible. She watched Star Wars Episode II and thought it was just as bad. After this, she concluded that all Star Wars movies are bad.

E) ***D*** ***I*** Luis took a creative writing class and got an A. Then he took a graphic novel class and got an A. He concludes that he will get an A in every English course.

F) ***D*** ***I*** The IRS tax code says that if you make more than $116,000 in income, then you cannot open a Roth IRA. Jackson earns $120,000 in income, so he concludes that he can't open a Roth IRA.

G) ***D*** ***I*** When he was in high school, Scott's parents told him that if he got home after 10pm, he was going to be grounded for 2 weeks. When he looked at his watch one Friday night during a rousing math battle, it was 10:15pm and he concluded that he was going to be grounded when he got home. *[So he stayed out later because, really, it just doesn't matter and he might as well just keep factoring those polynomials.]*

EXPLORE! Determine the type of reasoning used in these examples and decide if the conclusion is guaranteed.

	Statements	Conclusion	Type of Reasoning	Conclusion Guaranteed?
A) **	I ate at McDonald's and got sick. Then I ate at Wendy's and got sick.	Fast food makes me sick.	Deductive Inductive	Yes No
B) **(L)**	The Commutative Property of Multiplication is true.	$7 \times 6 = 6 \times 7$	Deductive Inductive	Yes No
C) **(R)**	All dogs are mammals. Fido is a dog.	Fido is a mammal.	Deductive Inductive	Yes No
D) **(L)**	Star Wars made a lot of money. Avatar made a lot of money.	Sci-fi movies make a lot of money, so John Carter will do well.	Deductive Inductive	Yes No
E) **(R)**	Doctors know anatomy. Sean is a doctor.	Sean knows anatomy.	Deductive Inductive	Yes No
F) **	A bullet fired in the air falls back down. A football thrown up in the air falls back down.	What goes up, must come down.	Deductive Inductive	Yes No

For Love of the Math: *It is important here to know that inductive reasoning, while the conclusion is not guaranteed, is still extremely important in math and science. Often, the inductive reasoning allows us to see patterns and begin to make a conjecture (educated guess) which we can explore to see if it is true or not. The* ***Scientific Method*** *uses this idea as a starting point:*
① Observe and Question,
② Create Hypothesis,
③ Design an Experiment to Determine if Hypothesis is True,
④ Analyze Results of the Experiment,
⑤ Revise, and then ⑥ Make a Conclusion.

In statistics, we use a similar process to determine what may be true about a set of data.

2.4: Formal Logic – Statements and Quantifiers.

To be more precise, we'll make sure that we know some of the key pieces that form a basis for logical reasoning.

Logic is based on **propositions** (or **statements**) – a sentence which is either true or false, but not both. Here are some examples along with truth values.

EXPLORE (1)! Determine whether the following are statements, and then the truth value.

	Sentence	Proposition	Truth Value
A) **	George Washington was the first president.	Yes No	True False
B) **	Where are you going?	Yes No	True False
C) **(L)**	This sentence is false.	Yes No	True False
D) **(R)**	Abe Lincoln was the fifth president.	Yes No	True False
E) **(L)**	$5 + 8 = 13$	Yes No	True False
F) **(R)**	$79 < 45$	Yes No	True False
G) **(L)**	The sky is cloudy today.	Yes No	True False
H) **(R)**	Scott Fallstrom is tall.	Yes No	True False
I) **(L)**	All carpenters are men.	Yes No	True False
J) **(R)**	Just do it!	Yes No	True False

With these statements, we may have some of these apply to no cases, some cases, or possibly all cases. Since these involve a quantity of statements, the words we use are called **quantifiers**. The quantifiers we use in this class are: All, Some, and None. **All** means every single one, **None** means not even one, and **Some** means there is at least one.

EXPLORE (2)! Determine whether the following quantified statements are true or false. Explain your answer.

A) ** All girls like Barbie.

B) ** Some cheese is yellow.

C) **(R)** No cats are green.

D) **(L)** Some cars are not dirty.

E) **(L)** All men are good at math.

F) **(R)** Some MiraCosta students are over 40 years old.

If a statement has one truth value and we would like the opposite truth value, we call that the **negation** of the statement.

Example 1: Find the negation of "I am happy."

The negation of "I am happy" is "I am not happy."

NOTE: Be careful when we take the negation of a statement that we don't change the meaning of the statement. Do not change "I am happy" to be "I am sad"… while happy and sad are often thought of as opposites, you could be not happy and also be not sad.

EXPLORE (3)! Write the negations of these statements.

A) ** The sky is blue.

B) It is 65 degrees outside.

C) **(L)** You sank my battleship.

D) **(R)** Donald Trump is a democrat.

The negation of quantified statements is a bit more challenging, but still doable. Finding a negation of a quantified statement requires us to think of what it would take to prove the statement false.

Example 2: Find the negation of "All college students make at least $50,000 per year."

Think about what it would take to prove this statement wrong. What if we found a student making $75,000 per year – would that prove it wrong? Nope. That person would actually be an example supporting the statement. We need to find one student (at least one) who makes less than $50,000 per year… if we can do that, then we have proven the statement wrong.

So the negation is: "Some college students make less than $50,000 per year."

Example 3: Find the negation of "Some puppies are brown."

Think about what it would take to prove this statement wrong. We can't find one white puppy to show that the statement is wrong because this statement doesn't say "All puppies are brown." For us to prove this statement is wrong, we actually need to get all puppies together and look at the color. If any puppy is brown, then the statement would be true. So for the statement to be false, we need to show that no puppy is brown… not even one. So the negation is: "All puppies are not brown" or "No puppies are brown." Either of these would work fine.

EXPLORE (4)! Determine the negation of the following quantified statements.

A) ** All girls like Barbie.

B) ** Some cheese is yellow.

C) **(L)** No cats are green.

D) **(R)** Some cars are not dirty.

Summarizing the negation of quantified statements along with equivalent statements:

General Quantified Statement	Negation	Equivalent (to the original)
All H are Q.	Some H are not Q.	No H are not Q.
Some H are Q.	No H are Q … or … All H are not Q.	At least one H is Q.
No H are Q.	Some H are Q.	All H are not Q.

EXPLORE (5)! Consider the following situation.

- Collection P has 3 movies: *Casablanca*, *Catching Amy*, and *Clear and Present Danger*.
- Collection Q has 4 movies: *Firefox*, *Fantastic Four*, *Dogma*, and *Assassins*.
- Collection R has 2 movies: *Smurfs* and *Entrapment*.
- Collection S has 1 movie: *Regarding Henry*.

Determine which of the following statements are true – circle your value:

	Statement	Truth Value
A) **	Some of the movies in collection P start with "C."	T F
B) **	Some of the movies in collection S don't start with "R."	T F
C) **	All of the movies in collection R start with "S."	T F
D) **(L)**	Some of the movies in collection Q start with "F."	T F
E) **(R)**	Some of the movies in collection R start with "P."	T F
F) **(L)**	Some of the movies in collection R don't start with "P."	T F
G) **(R)**	Some of the movies in collection S start with "R."	T F
H) **(L)**	All of the movies in collection P start with "C."	T F
I) **(R)**	All of the movies in collection S start with "R."	T F
J) **(L)**	None of the movies in collection R start with "P."	T F
K) **(R)**	None of the movies in collection Q start with "D."	T F

L) Look at statements (B) and (G) above. (B) was false, but (G) was true. Think about (B) and determine what it means to be false in this example? Explain.

M) Explain why the statements "Some dogs are yellow" and "Some dogs are not yellow" are **<u>not</u>** equivalent?

Symbolically, we sometimes shorten the statements to a single lowercase letter and use the symbol $\sim p$ to represent the negation of *p*. So we write it this way:

- *p*: I am happy

When we have a statement like "It is not sunny" and we want to symbolically represent it, there are a few ways to do that.

Method 1: keep the symbolic statement in the positive, then use the ~ symbol.

- *p*: It is sunny.
- ~ *p*: It is not sunny.

Method 2: keep the symbolic statement as it is written.

- *p*: It is not sunny.
- ~ *p*: It is sunny.

When a problem is given, you are in charge of picking the symbolic representation that you want to use. Either method works just fine!

<u>Interactive Example:</u> In order to consider all possibilities, we often use a "**<u>Truth Table</u>**" which is a way of considering all different outcomes for a statement.

	p	$\sim p$
A)		
B)		

Write the statements you are considering out first. In this case, we'll consider *p* and ~ *p*. Since each statement can be either true or false, we will use T and F to represent those options, putting one in row (a) and one in row (b).

	p	$\sim p$
A)	T	
B)	F	

Now, to complete the truth table, we put in the truth values for the other statement ~ *p*. Remember that if a statement was true, the negation would be false. *<u>Fill in the appropriate values above to complete the truth table.</u>*

<u>EXPLORE (6)!</u> Complete the rest of the rows of the truth table to determine the options for $\sim(\sim p)$

	p	$\sim p$	$\sim(\sim p)$
A)	T		
B)	F		

Based on the truth table, we saw that *p* had the same truth values as $\sim(\sim p)$ in every situation. Because each is the same, we call these two **<u>logically equivalent statements</u>** and write $\sim(\sim p) \equiv p$.

2.5: Formal Logic – Operators (Truth Tables).

Now that we have the idea of a statement, we will create some operations on these statements. For this course, we will use 4 main operations (in addition to negation):

- Disjunction (or)
- Conjunction (and)
- Conditional (if-then)
- Bi-Conditional (if and only if)

As we put some simple statements together, they become more complicated – we call these **compound statements**. Since they involve more than one simple statement, we will need additional rows in the truth table. There are 2 choices for *p* (T and F), but also 2 choices for *q* (T and F). We need to list all possible combinations.

	p	q
A)	T	T
B)	F	T
C)		
D)		

Interactive Example: Above, we considered *q* being true, and listed both options for *p*. Fill in the other possibilities in the empty spaces.

The first compound statement we encounter is formally called a **conjunction**, which represents simple statements connected with the word 'and.' When we write these symbolically, the symbol for "*p* and *q*" looks like $p \wedge q$. The way to think about conjunction is that in order for $p \wedge q$ to be true, **both *p* and *q* must be true.**

	p	q	$p \wedge q$
A)	T	T	T
B)	F	T	F
C)	T	F	F
D)	F	F	F

EXPLORE (1)! Determine if the truth values of the statements listed.

	p	q	$p \wedge q$	$\sim p$	$\sim q$
A)	T	F			
B)	F	F			
C)	T	T			
D)	F	T			

The second compound statement we encounter is formally called a **<u>disjunction</u>**, which represents simple statements connected with the word 'or.' When we write these symbolically, the symbol for "*p* or *q*" looks like $p \vee q$. NOTE: the 'or' statement here refers to the *inclusive-or*, not the *exclusive-or*. The inclusive-or means that **one or the other or both must be true**. Another way to phrase this is that at least one of the simple statements must be true for a disjunction to be true.

	p	q	$p \vee q$
A)	T	T	T
B)	F	T	T
C)	T	F	T
D)	F	F	F

EXPLORE (2)! Determine if the truth values of the statements listed.

	p	q	$p \vee q$	$p \wedge q$	$\sim q$
A)	F	F			
B)	F	T			
C)	T	F			
D)	T	T			

Now we can put these operators together: conjunction, disjunction, and negation.

EXPLORE (3)! Determine if the truth values of the statements listed.

	p	q	$\sim p$	$\sim q$	$p \vee (\sim q)$	$(\sim p) \wedge q$	$(\sim p) \wedge (\sim q)$
A)	T	F					
B)	F	T					
C)	F	F					
D)	T	T					

The next compound statement we encounter is formally called a **conditional**, which represents simple statements connected with the words 'if-then.' When we write these symbolically, the symbol for "if *p*, then *q*" looks like $p \to q$. The first portion of a conditional is called the **condition** (or antecedent) while the second part is called the **conclusion** (or consequent). In order to determine when this one is true or false, let's consider an example.

Interactive Example: Determine the truth values of a realistic conditional statement. Try to determine when any statement breaks this rule: "If you earn an 85%, then I will assign you a grade of B."

	Student Earns	**Grade Assigned**	**Truth or Lie**
A)	85%	B	Truth Lie
B)	82%	B	Truth Lie
C)	93%	A	Truth Lie
D)	85%	C	Truth Lie

What we find from this example is that a conditional statement is only *false* when you meet the condition but fail the conclusion. The truth table for a conditional statement looks like this:

	p	q	$p \to q$
A)	T	T	T
B)	F	T	T
C)	T	F	F
D)	F	F	T

EXPLORE (4)! Determine if the truth values of the statements listed.

	p	q	$\sim p$	$\sim q$	$q \to p$	$(\sim p) \to (\sim q)$	$(\sim q) \to (\sim p)$
A)	T	T					
B)	F	T					
C)	T	F					
D)	F	F					

We see conditionals often written different ways, sometimes using the negation. When they are written in some special ways, we have specific names.

- Conditional: $p \to q$
- Converse: $q \to p$
- Inverse: $(\sim p) \to (\sim q)$
- Contrapositive: $(\sim q) \to (\sim p)$

Two statements are **logically equivalent** if they have exactly the same truth values. Based on the truth tables above, which of these are logically equivalent?

Example: Find an equivalent expression to "If you build it, they will come."

Because the contrapositive is logically equivalent, we can write: "If they won't come, then you didn't build it." Notice how we negate both and then change the order – a process known as "taking the contrapositive."

EXPLORE (5)! Find an equivalent expression to the given statement.

A) ** If Josh is not late, then we will go to Star Wars.

B) ** You are happy, if it is sunny.

C) **(L)** If dogs are brown, then aliens do not exist.

D) **(R)** If Macy is not angry, then Jason will not be grounded.

NEGATING OPERATORS

Now that we have our major operators (conjunction, disjunction, and conditional), let's determine how to work with the negation of each of them.

	p	q	$p \wedge q$	$p \vee q$	$\sim(p \wedge q)$	$\sim(p \vee q)$
A)	T	T	T	T		
B)	F	T	F	T		
C)	T	F	F	T		
D)	F	F	F	F		

Earlier in this section, we had some expressions that were exactly the as these, leading to this:

- $\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$
- $\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$

These are known as **De Morgan's Laws**, after the British mathematician Augustus De Morgan.

If we try to negate a conditional, it is a bit different. There is only one way to show a conditional statement is false: when you meet the condition but fail the conclusion. This gives us an idea of how to prove a conditional statement false.

Example: Find what would make this false: "If you go to school, then you are happy."

To show that this is false, we would need to find someone who goes to school but is not happy. Just one.

	p	q	$\sim q$	$p \to q$	$\sim(p \to q)$	$p \wedge (\sim q)$**
A)	T	T	F	T		
B)	F	T	F	T		
C)	T	F	T	F		
D)	F	F	T	T		

Based on the information in the table, we can now see: $\sim(p \to q) \equiv p \wedge (\sim q)$.

Summary of the symbolic negations:

- Negation of negation: $\sim(\sim p) \equiv p$
- Negation of conjunction: $\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$
- Negation of disjunction: $\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$
- Negation of conditional: $\sim(p \to q) \equiv p \wedge (\sim q)$

EXPLORE (6)! Use your negation knowledge to negate the following statements:

A) ** I am hot if it is 90 degrees.

B) ** It is rainy and I am not happy.

C) **(L)** If Jo is pretty, then the sky is blue.

D) **(R)** No news is good news.

E) **(R)** Logs are not heavy and juice is cold.

F) **(L)** All dogs run fast.

G) **(R)** Some books are in French.

H) **(L)** I get an A or I get a B.

EXPLORE (7)! Determine whether the original and new statements are equivalent, negations, or neither.

	Original	New	Determination
A) **	$p \to q$	$(\sim q) \to (\sim p)$	Equivalent Negation Neither
B) **	$p \vee q$	$p \wedge q$	Equivalent Negation Neither
C) **(R)**	$p \vee q$	$q \vee p$	Equivalent Negation Neither
D) **(R)**	$p \wedge q$	$\sim p \vee \sim q$	Equivalent Negation Neither
E) **(L)**	$p \to q$	$q \to p$	Equivalent Negation Neither
F) **(R)**	$q \to p$	$(\sim p) \to (\sim q)$	Equivalent Negation Neither
G) **(L)**	$q \to p$	$q \wedge \sim p$	Equivalent Negation Neither
H) **(L)**	$p \to q$	$q \leftarrow p$	Equivalent Negation Neither

EXPLORE (8)! If p is true, q is true, and r is false, determine the whether the compound statement is true (T), false (F), or if you can't tell because you need more information (N). The statement w is not known to be true or false, so you may be able to still make a conclusion without knowing its value.

	Statement	Determination	Explanation
I) **	$p \vee r$	T F N	
J) **	$q \to w$	T F N	
K) **(L)**	$p \vee w$	T F N	
L) **(L)**	$w \wedge q$	T F N	
M) **(R)**	$p \wedge r$	T F N	
N) **(L)**	$p \to q$	T F N	
O) **(R)**	$q \to r$	T F N	
P) **(R)**	$r \to w$	T F N	

2.6: Expanded Conditionals and Bi-Conditionals.

Conditionals are great, but there are many ways to write them. We know the "if-then" and could think about reversing these to be "then-if," but what about another.

Interactive Example: Write this statement in the form of an if-then statement: "You get parole only if you have good behavior."

What would happen if you don't have good behavior? Then you wouldn't get parole! Let's put this in symbols and see if we can make it look like one of our other statements.

- *p*: you get parole
- *q*: you have good behavior.

So the original statement "You get parole only if you have good behavior" is "*p* only if *q*"

Now we know that if you don't have good behavior, then you won't get parole. We could write this as the following: $(\sim q) \rightarrow (\sim p)$.

The symbolic contrapositive of $(\sim q) \rightarrow (\sim p)$ is ___________________.

Write an English statement for the contrapositive of $(\sim q) \rightarrow (\sim p)$:

So after all this, it seems that "*p* only if *q*" is logically equivalent to $p \rightarrow q$. Recall that "*p* if *q*" is logically equivalent to $q \rightarrow p$, or $p \leftarrow q$ if you prefer. These are not equivalent!

EXPLORE (1)! Write the following symbolically as if-then statements.

- *p*: you are happy
- *q*: it is sunny
- *r*: roses are blue

A) ** You are happy only if it is sunny.

B) **(R)** It is sunny if roses are blue.

C) **(R)** Roses are not blue if you are not happy.

D) **(L)** If you are not happy, then it is sunny.

E) **(L)** Roses are blue only if it is sunny.

This brings us to the last piece of our logic puzzle. If we have some if-then statements like $q \to p$ and $p \to q$, we could think of these as "*p*, if *q*" and "*p* only if *q*" based on the last pages. Here are the truth tables for those statements:

	p	q	$q \to p$	$p \to q$
A)	T	T	T	T
B)	F	T	F	T
C)	T	F	T	F
D)	F	F	T	T

Now let's create a new statement composed of the last two: $(q \to p) \wedge (p \to q)$. What word does the symbol $\wedge$ represent?

<u>Interactive Example:</u> Let's create the truth table for this new one: $(q \to p) \wedge (p \to q)$.

	p	q	$q \to p$	$p \to q$	$(q \to p) \wedge (p \to q)$
A)	T	T	T	T	
B)	F	T	F	T	
C)	T	F	T	F	
D)	F	F	T	T	

Because the wording of $(q \to p) \wedge (p \to q)$ is "*p* if *q* and *p* only if *q*" we could combine this group of two conditionals into one new statement formally called a <u>**bi-conditional**</u>, which represents simple statements connected with the words 'if-and-only-if' which can be abbreviated as 'iff.' When we write these symbolically, the symbol for "*p* iff *q*" looks like $p \leftrightarrow q$.

So $p \leftrightarrow q \equiv (q \to p) \wedge (p \to q)$.

Let's check that table one more time to see a pattern, something you could put into words. Look for a pattern!

	p	q	$q \to p$	$p \to q$	$p \leftrightarrow q$
A)	T	T	T	T	T
B)	F	T	F	T	F
C)	T	F	T	F	F
D)	F	F	T	T	T

In your words, how do you know when a bi-conditional is true, or when it is false?

<u>EXPLORE (2)!</u> Try a few bi-conditionals to determine when they are true and when they are false. Be able to put them in your words. The first one is done for you.

	Statement	Truth Value	Explanation
A) **	2 + 2 = 4 iff 3 + 5 = 7	T F	The first statement is true, but 3 + 5 = 7 is false. Since these have different truth values, the bi-conditional is false.
B) **	2 + 2 = 4 iff 3 + 5 = 8	T F	
C) **(R)**	A number is divisible by 5 iff the number ends with a 0 or 5.	T F	
D) **(L)**	A number is divisible by 2 iff the number ends with a 6.	T F	
E) **(R)**	You will be financially stable iff you work hard.	T F	
F) **(L)**	$(4\times 2=9)\leftrightarrow(5-3<1)$	T F	

One of the conditional statements can be true and the other false. When that happens, the bi-conditional statement is false. But this is a point of contention for many people – seeing one conditional to be true makes it seem that both must be true. Keep in mind that conditionals and bi-conditionals are different and that you may need to list out a conditional statement clearly to see it.

<u>WHAT'S THE POINT?</u>

Understanding logic and reasoning allows whole new worlds to open up, as these are used in all areas.

Examples:

- Statistics: If the p-value is less than 0.05, you must reject the null hypothesis.
- Government: If the President and Vice-President are incapacitated, then the next in line to be President would be the Speaker of the House.
- Television: If the ratings for "Blacklist" drop below 8.2, then it will be cancelled.
- Football: If the Chargers don't get a new stadium deal, then they will move to LA.
- Math 52: $a - b = c$ if and only if $c = b + a$.
- Games (pinochle): If hearts are led and you have a heart, then you must play the heart.

Conditional statements are often used when a bi-conditional should be used, and because many lack deep understanding of these ideas, students often apply bi-conditional reasoning to conditional statements… which creates problems. Here is perhaps the most famous example:

Application to Sociology/Psychology (Wason Selection Test).

Here's the set up: You are shown 4 cards – each card has a number on one side and a color on the other side. You want to test the following statement and determine if it is true: "If a card shows an even number on one side, then its opposite side is red."

Here are the 4 cards that you are shown:

Card 1	Card 2	Card 3	Card 4
5	8	Red	Green

Which card (or cards) do you need to turn over to test the truth of this proposition? Explain your reasoning.

Visit this link to play the game again: https://www.youtube.com/watch?v=qNBzwwLiOUc

EXPLORE (3)! Now we'll model this with a slight twist. You are now the person checking IDs at a party where everyone is drinking something. There is one rule you must follow: If someone is under 21, they can't be drinking alcohol. You see 4 people with the following (limited) information:

Person 1	Person 2	Person 3	Person 4
Age = 35	Age = 18	Drinking Mt. Dew	Drinking Alcohol

Which of these 4 people do you need to get more information about? Explain why below, then compare with someone else in class.

As we move forward, conditional statements lead us through arguments and determining whether someone is making a solid argument. Knowing how these fit together will give you the opportunity to make your own arguments to convince others in verbal or written form. When you write a conditional, it is strongly encouraged that you also write the contrapositive so you can see another form of the same statement! Rewriting equivalent statements that are compound is a bit more challenging, but certainly doable.

Example: Find an equivalent statement to "If you have a permission slip from your parents or are over 18, then you can go on the field trip."

It may help to write in symbolic form, so first we can define our simple statements.

- p: you have a permission slip from your parents
- q: you are over 18
- r: you can go on the field trip.

Now write out the statement in symbols: $(p \vee q) \to r$. Since this is a conditional statement, we can apply the contrapositive which reverses the order and negates both statements: $(p \vee q) \to r \equiv \sim r \to \sim (p \vee q)$

Lastly, we use De Morgan's law to rewrite the negation of the "or" statement: $\sim (p \vee q) \equiv (\sim p) \wedge (\sim q)$.

So $(p \vee q) \to r \equiv \sim r \to \sim (p \vee q) \equiv \sim r \to ((\sim p) \wedge (\sim q))$. When finished, put the statement back into words. An equivalent statement to "If have a permission slip from your parents *or* are over 18, then you can go on the field trip" is this: "If you can't go on the field trip, then you don't have a permission slip from your parents *and* you are not over 18."

EXPLORE (4)! Try one on your own. Find an equivalent statement to these conditionals.

A) ** If you love me, then you won't leave and you'll buy me a car.

B) If you are filing as single and make more than $116,000, then you do not qualify for a Roth IRA.

C) If you're a good person and you like kittens, then you will rescue a kitten or will donate money to the animal shelter.

Symbolic reasoning summary: For these exercises, you need to determine whether the statement is true or false based on the clues.

Example: Determine the truth value of $(p \vee q) \to r$ if we know that *p* is true, *q* is false, and *r* is false.

A good way of doing this is to substitute in the values that you know to rewrite the expression (like we did with variables and numbers in Math 52). $(p \vee q) \to r \equiv (T \vee F) \to F$.

Now go through and do one part at a time. Start with the piece in parenthesis: $(T \vee F)$. This is an "or" statement, which requires at least one simple statement to be true. Since one of them is true, then $(T \vee F) \equiv T$. Rewrite our expression to simplify it: $(p \vee q) \to r \equiv (T \vee F) \to F \equiv T \to F$.

The last piece is to determine the value of $T \to F$. This is a conditional statement, and the only time a conditional statement is false is when we meet the condition and fail the conclusion. That's exactly what happened here, so this is false. $T \to F \equiv F$. So $(p \vee q) \to r$ is false!

EXPLORE (5)! Determine the truth values of these statements if you know that *p* is true, *q* is false, and *r* is false. If you see any other letter for a statement, it could be true or false… we don't know. Your answer could be true, false, or need more information.

	Statement	Determination	Explanation
A) **	$p \wedge (\sim q)$	T F N	
B) **	$p \vee w$	T F N	
C) **	$w \to q$	T F N	
D) **(R)**	$(p \wedge q) \to w$	T F N	
E) **(R)**	$(q \wedge r) \to p$	T F N	
F) **(L)**	$p \to (q \vee r)$	T F N	
G) **(L)**	$(\sim r) \wedge w$	T F N	

<u>2.7: Basic Arguments – Using Logic</u>

As stated earlier in the unit, an argument requires a number of <u>**premises**</u> (facts or assumptions) which are followed by a <u>**conclusion**</u> (point of the argument). The premises are used as justification for a conclusion. A conclusion which is correctly supported by the premises is known as a <u>**valid argument**</u>, while a <u>**fallacy**</u> is a deceptive argument can sound good but is not well supported by the premises.

Here, we won't focus on fallacies but will focus on the argument itself. An argument is essentially a large conditional statement. To illustrate this, here's an example.

<u>Example 1:</u> Determine if the following argument is valid.

1. All men are mortal.
2. <u>John Smith is a man.</u>
c. John Smith is mortal.

There are 2 premises and one conclusion. The argument in symbolic form is: $(1 \wedge 2) \rightarrow c$. In order for the argument to be valid, we need this conditional statement to be always true. If there is ever a time, <u>*even just one time*</u>, when this conditional statement is false, then it is an <u>**invalid argument**</u>. Another way to think of this is to say that the conclusion must follow from the premises… if the premises are true, then the conclusion <u>*must*</u> be true in order to be valid.

Look at the argument – if we <u>*assume*</u> that 1 and 2 are both true, then does the conclusion have to follow? YES! If all men are mortal, and if John Smith is a man, then John Smith must be mortal. This is valid.

<u>Example 2:</u> Determine if the following argument is valid. (Hint: rewrite the "all" as "if-then", then also write the contrapositive)

1. All dogs are yellow. 2. Flippy is a dog. c. Flippy is yellow.

All dogs are yellow is equivalent to "if it is a dog then it is yellow." That is equivalent to "if it is not yellow, then it is not a dog" by the contrapositive. Assume both (1) and (2) are true. Does the conclusion (c) have to follow? YES! So this is valid!

<u>Example 3:</u> Determine if the following argument is valid. (Hint: rewrite the "all" as "if-then", then also write the contrapositive)

1. All dogs are yellow. 2. Chipper is yellow. c. Chipper is a dog.

All dogs are yellow is equivalent to "if it is a dog then it is yellow." That is equivalent to "if it is not yellow, then it is not a dog" by the contrapositive. Assume both (1) and (2) are true. Does the conclusion (c) have to follow? None of our premises describe what happens if something <u>*actually is*</u> yellow. (1) says that all dogs are yellow, but doesn't say that everything that is yellow is a dog. It is very possible to have something that is yellow but not a dog; that means the conclusion isn't necessarily true (think of a yellow cat or a yellow pencil). This argument is invalid.

These do take some practice, but it is good to get our minds working!

<u>EXPLORE (1)!</u> Determine if the following arguments are valid. Explain your reasoning.

A) **

1. If you are a gambler, then you are not financially stable.
2. <u>Sean isn't financially stable.</u>

c. Sean is a gambler.

B) **(L)**

1. If you have a college degree, then you are not lazy.
2. <u>Marsha has a college degree.</u>

c. Marsha is not lazy.

C) **(L)**

1. If you have a college degree, then you are not lazy.
2. <u>Shannon is lazy.</u>

c. Shannon does not have a college degree.

D) **(R)**

1. If you have a college degree, then you are not lazy.
2. <u>Beth is not lazy.</u>

c. Beth has a college degree.

E) **(R)**

1. All licensed drivers have insurance.
2. If you have insurance, then you obey the law.
3. <u>You obey the law.</u>

c. You are a licensed driver.

F) **
1. All licensed drivers have insurance.
2. If you have insurance, then you obey the law.
3. <u>You don't obey the law.</u>
c. You are not a licensed driver.

Sometimes, it is up to us to determine the conclusion to an argument. Let's practice some of these.

<u>EXPLORE (2)!</u> Determine a conclusion that would make the argument valid, if possible.

A) **
1. If you are a gambler, then you are not financially stable.
2. <u>Hollis is financially stable.</u>
c.

B) **(L)**
1. If a defendant is innocent, then he does not go to jail.
2. <u>Podric went to jail</u>
c.

C) **(R)**
1. If you listen to speed metal, then you don't go to heaven.
2. If you are a good person, then you go to heaven.
3. <u>Rachel is a good person.</u>
c.

D) **(L)**
1. If you get an audit, then you didn't fill out your tax forms correctly.
2. If you win the lottery, then you will get an audit.
3. <u>Tyson filled out his tax forms correctly.</u>
c.

E) **(R)**
1. All doctors are men.
2. <u>My mother is a doctor.</u>
c.

<u>EXPLORE (3)!</u> Determine whether there is a problem with the person's thinking. Be able to explain your reasoning.

A) Louis read the IRS guideline: "If your income on line 37 is more than $100,000 and you are filing as single, then you must fill out form J." His income is $150,000 and he is married filing jointly, so he didn't fill out form J. He was frustrated when he was audited for not filling out his taxes properly. Does his complaint hold water?

B) **(L)** John's mom told him "If you get home after 10pm, then you are grounded." John got home at 9:30pm and was grounded. He was really ticked off because he said that she lied to him. Did she?

C) **(R)** Marcia told her daughter: "If you get home before 10pm, then I will give back your cell phone." Her daughter got home at 9:45pm, but her mom didn't give back the cell phone. Does the daughter have a right to be angry?

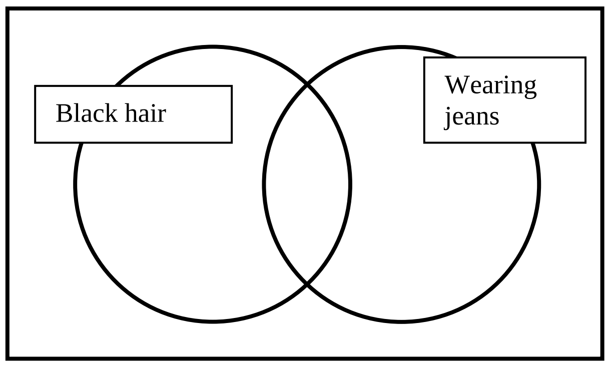
2.8: Arguments – Using Venn Diagrams

For many people, the arguments we've been talking about are somewhat fun. This section shows a way to analyze arguments (including quantified statements) in a visual way.

In order to visually represent this argument, we will use shapes, typically circles, to represent objects. The large outer box will represent all people or all objects of one type. We will label each circle to describe a group of people or objects. These pictures are called **Venn diagrams**, named after English mathematician John Venn.

Interactive Example (1): Determine where to put each person in the Venn diagram, if possible.

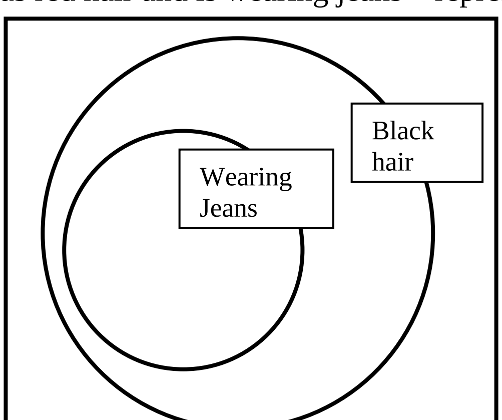
A) Madira has black hair and is wearing jeans – represent her with "M."
B) Scott has sandy blonde hair and is wearing shorts – represent him with "S."
C) Osha has black hair and is wearing a skirt – represent her with "O."
D) Pete has red hair and is wearing jeans – represent him with "P."



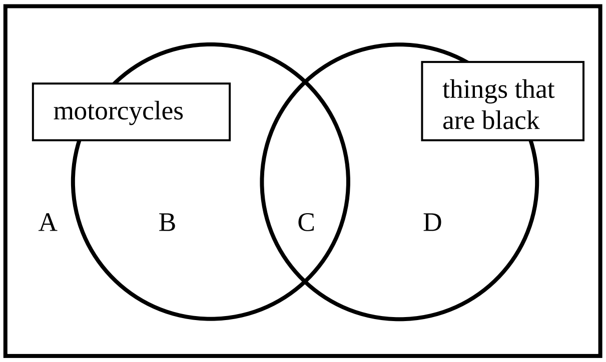
Describe this Venn diagram in words: "Some people with black hair wear jeans."

Interactive Example (2): Determine where to put each person in the Venn diagram, if possible.

A) Madira has black hair and is wearing jeans – represent her with "M."
B) Scott has sandy blonde hair and is wearing shorts – represent him with "S."
C) Osha has black hair and is wearing a skirt – represent her with "O."
D) Pete has red hair and is wearing jeans – represent him with "P."



Describe the Venn diagram in words.
Here's an example that would represent the statement "Some motorcycles are black."



<u>EXPLORE (1)!</u> Based on the picture, determine <u>*in words*</u> what each letter region represents.

A) **(L)**

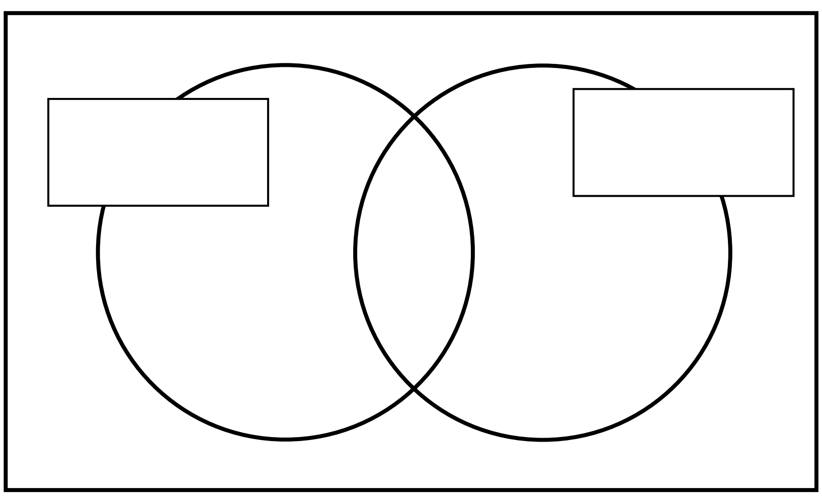
C) **(R)**

B) **(L)**

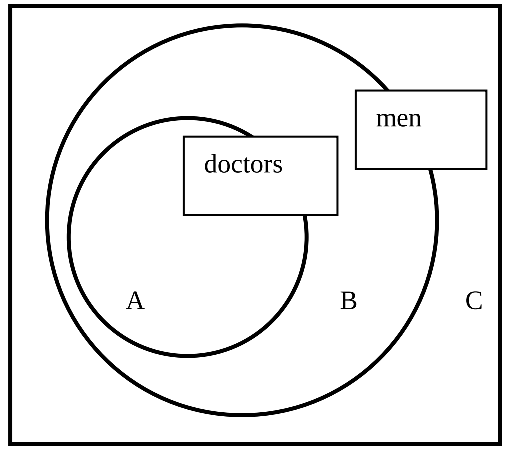
D) **(R)**

E) In the Venn diagram above, is there exactly one place for a motorcycle, or is there more than one? Explain.

<u>EXPLORE (2)!</u> Label the circles in a Venn diagram that represents "Some cheese is stinky." Then put an "X" in the spot that would represent a <u>stinky perfume</u> and "Y" in the spot representing a <u>stinky cheese</u>.



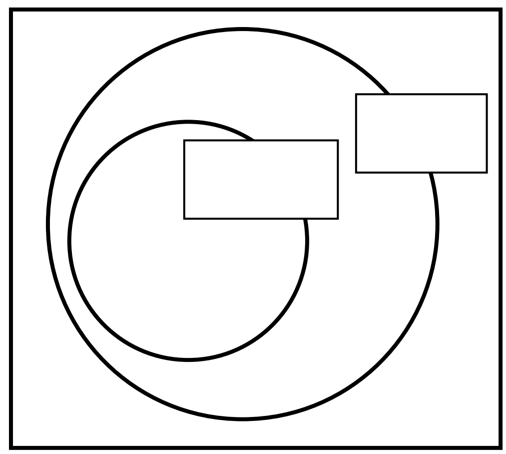
Here’s an example that would represent the statement “All doctors are men” or “If you are a doctor, then you are a man.”



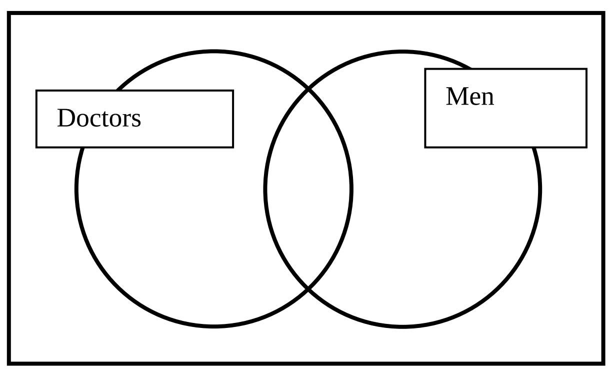
EXPLORE (3)! Based on the picture, determine in words what each letter region represents.

A) **   B) **(R)**   C) **(L)**

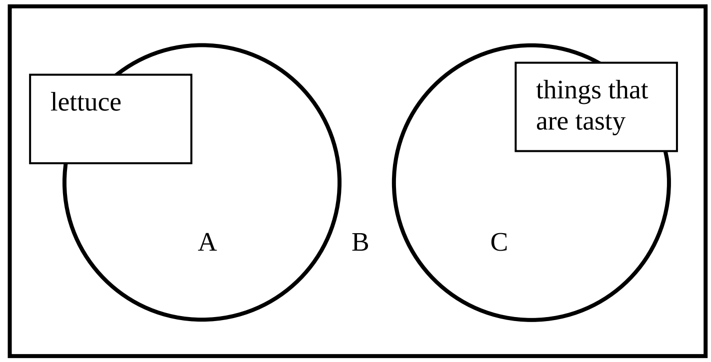
EXPLORE (4)! Label a Venn diagram that represents “If you are guilty, then you go to jail.” Then put an “X” in the spot that would represent an innocent person who is in jail and “Y” in the spot that would represent an innocent person who is not in jail.



NOTE: You could try to put the statement “All doctors are men” into a different looking Venn diagram. However, there would be a region here that is completely empty. Which region must be empty based on the statement? Shade that region and explain why.



Here's an example that would represent the statement "No lettuce is tasty" or "If it is lettuce, then it is not tasty."



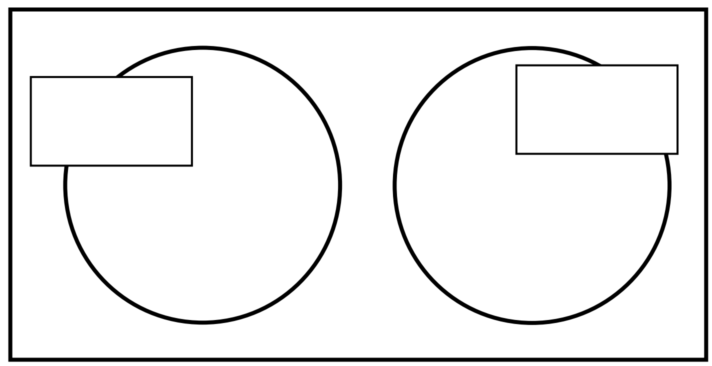
<u>EXPLORE (5)!</u> Based on the picture, determine in words what each letter region represents.

A) **

B) **(R)**

C) **(L)**

<u>EXPLORE (6)!</u> Label a Venn diagram that represents "No happy people drink Mt. Dew." Then put an "X" in the spot that would represent a happy person and "Y" in the spot that would represent an unhappy person. There may be more than one answer.

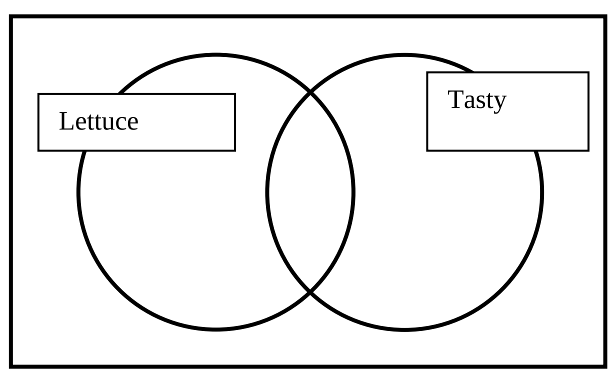


<u>EXPLORE (7)!</u> Convert the quantified statement "No happy people drink Mt. Dew" into a conditional statement (if-then). There are two ways to write this.

A) **(R)** If you are happy, …

B) **(L)** If you drink Mt. Dew, …

<u>NOTE:</u> You could try to put the statement "No lettuce is tasty" into a different looking Venn diagram. However, there would be a region here that is completely empty. Which region must be empty based on the statement? Shade that region and explain why.



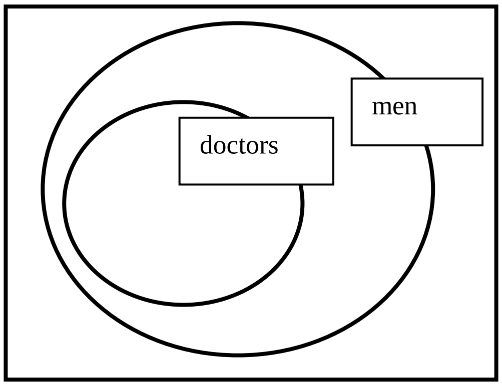
Now let's see how to use these Venn diagrams to analyze an argument, and see if it is valid.

<u>Example:</u> Determine if the argument is valid using a Venn diagram.

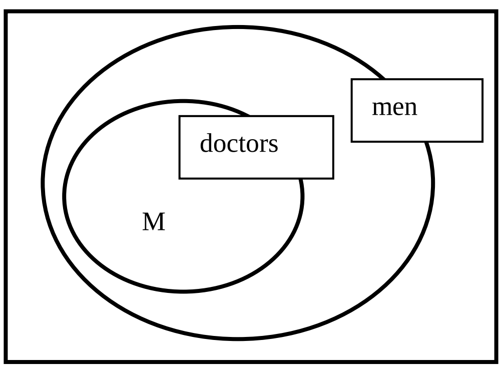
1. All doctors are men. 2. My mother is a doctor. c. My mother is a man.

Take each statement and integrate into a Venn diagram – remember, we assume that the premises are true and try to see if the conclusion must follow.

The first premise is that "All doctors are men" so make this into a Venn diagram.



The second premise is that "My mother is a doctor" so if this is true, put an 'M' in the Venn diagram to show where mother is located.



Now, from this picture, is it guaranteed that M is a man? Yes, M must be in the men circle, so the conclusion must follow from the premises. This argument is valid!

<u>For Love of the Math:</u> *We intentionally chose this argument because it is a little silly… there is no way my mother can be a man. However, if the premises are true, then the conclusion follows, so it is a valid argument.*

When the premises are true <u>and</u> the argument is valid, we call this a **<u>sound</u>** argument. When the premises are not true (and we know that "All doctors are men" is not a true statement),but the argument is valid, we call the argument **<u>not sound</u>**. So arguments can be (1) valid and sound, (2) valid and not sound, or (3) invalid. If it is invalid, we don't even describe whether it is sound!

EXPLORE (8)! Determine if the argument is valid using a Venn diagram. Remember, if there is any way that the conclusion doesn't follow, then the argument is invalid! Also, determine if this argument is sound.

1. Some hippies ride bicycles.
2. Jonathan rides a bicycle.

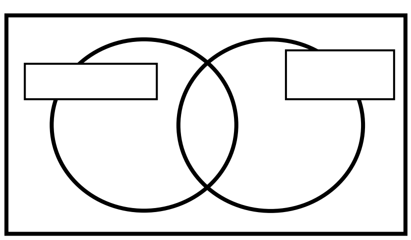
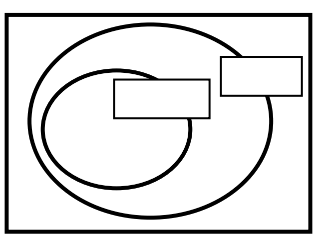
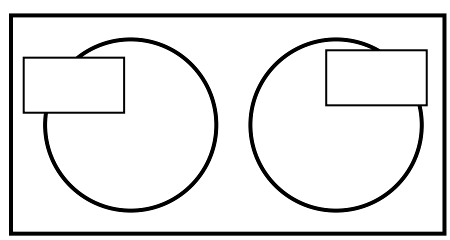
c. Jonathan is a hippie.

EXPLORE (9)! Determine if the argument is valid using a Venn diagram. Remember, if there is any way that the conclusion doesn't follow, then the argument is invalid! Also, determine if this argument is sound.

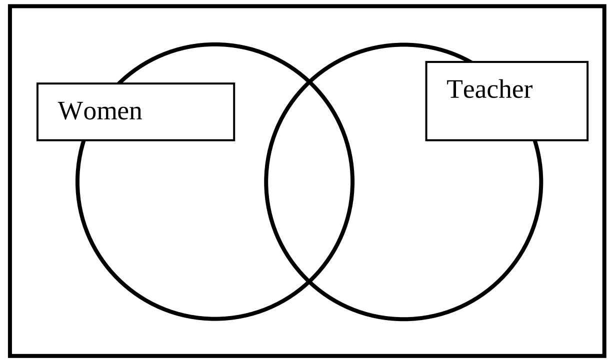
1. No fast food is healthy.
2. Salads are healthy.

c. Salads are not fast food.

What are the differences between the *All*, *Some*, and *None* Venn diagrams and which picture matches which wording?

EXPLORE (10)! Take a look at the Venn diagram. Determine whether you need a circle to represent men or not? Explain.



EXPLORE (11)! Determine if the argument is valid using a Venn diagram. Remember, if there is any way that the conclusion doesn't follow, then the argument is invalid! Also, determine if this argument is sound.

1. If you are innocent, then you don't go to jail.
2. Sam is innocent.
3. If you go to jail, then you are a bad person.

c. Sam is a bad person.

2.9: Learning Deductive Reasoning with Mastermind

Game #2: **Mastermind. http://www.web-games-online.com/mastermind** (this one has 8 colors, not 6)

http://www.archimedes-lab.org/mastermind.html (this has 6 colors only)

You could also go to the Google Play store and find "Guess the code" by Optime Software. This is an excellent version of Mastermind and it can be played on your phone or i-Pad.

This game uses colored pegs – one person or a computer makes a "code" that is comprised of these pegs in order. We'll play this game with smaller than normal game rules to start.

- There are 6 different colors for code-pegs in the standard game: We will use the colors Red (R), Orange (O), Yellow (Y), Green (G), Blue (B), Purple (P).
- There are 4 code-pegs put in order and hidden – this forms the "code." Code-pegs could use a color more than once.
- With each guess by the player, the code-maker gives clues.
    - White clue means that a peg is the correct color but not the correct location. (WH)
    - Black clue (or red clue depending on your version) means that a peg is both the correct color and the correct location. (BL)
    - These Black/White clues don't tell you which peg is the correct one.
- We will begin with some sample clues to show how the game is played.

Example #1: (Includes only two locations to start)

Guess	Location 1	Location 2	Clues
1	R	G	WH-WH
CODE	R G Y B	R G Y B	

What does the first guess tell us? Since both clues are white, it means that we have two chips with the correct color, but they are both in the wrong space. So we know that the first one should be G and the second should be R! That's the winning code.

EXPLORE (1)! (Includes only two locations to start) Find the code – share your explanation with someone else near you.

Guess	Location 1	Location 2	Clues
1	R	G	BL
2	B	Y	WH
3	G	G	
CODE	R G Y B	R G Y B	

Example #2: (Includes only three locations to start)

Guess	Location 1	Location 2	Location 3	Clues
1	R	G	B	WH-WH-WH
2	Y	B	R	BL-BL
CODE	R G Y B	R G Y B	R G Y B	

What does the first guess tell us? Since all clues are white, it means that we have the colors correct but the locations are wrong. That means the solution will have R-G-B in some order. The second guess tells us that there are 2 correct, but we already know that Y can't be one of the solution chips. So the B and R must be correct. Circle those in the "code" boxes, and then circle the correct color for the first choice. The winning code is G-B-R.

EXPLORE (2)! (Includes only three locations to start) Find the code – share your explanation with someone else near you.

Guess	Location 1	Location 2	Location 3	Clues
1	R	G	B	BL
2	G	B	Y	
CODE	R G Y B	R G Y B	R G Y B	

Example: (all 4 locations and 6 colors)

Guess	Location 1	Location 2	Location 3	Location 4	Clues
1	B	G	G	G	BL
2	P	R	Y	B	BL-BL-WH-WH
3					
CODE	R O Y G B P	R O Y G B P	R O Y G B P	R O Y G B P	

The 2^{nd} guess gives a lot of information: we know that the color code must have P-R-Y-B in some order. Two of those are in the right place, but two are not. While it seems that there are a lot of options, let's use the information in guess #1.

Notice that the first guess says one of the chips is in the right place. Since the code must have P-R-Y-B only, we know that the B chip must be correct. This information alone gives us enough to figure out the rest of the code. *Explain what the code must be and why*.

GROUP ACTIVITY: Grab a physical game board and play the game with your neighbor. Play with only 4 color pegs and no repetition of colors. Once one person wins, switch roles.

<u>EXPLORE (3)!</u> Determine information from these sample game boards. For these, use only 4 color pegs (R, G, Y, and B). Circle the correct solution – cross of those you know are not correct. Explain your reasoning with a sentence or two.

A) **

Guess	Location 1	Location 2	Location 3	Location 4	Clues
1	R	R	B	B	BL-BL
2	Y	Y	G	G	WH-WH
3	R	R	Y	Y	
CODE	R G Y B	R G Y B	R G Y B	R G Y B	

B) **(R)**

Guess	Location 1	Location 2	Location 3	Location 4	Clues
1	Y	G	B	G	BL-WH-WH-WH
2	G	B	Y	G	WH-WH-WH-WH
3	R	G	G	R	BL-BL
CODE	R G Y B	R G Y B	R G Y B	R G Y B	

C) **(L)**

Guess	Location 1	Location 2	Location 3	Location 4	Clues
1	G	B	R	R	BL-BL-WH-WH
2	B	R	Y	R	WH-WH-WH
3	R	Y	R	Y	BL-BL
CODE	R G Y B	R G Y B	R G Y B	R G Y B	

D) **(R)**

Guess	Location 1	Location 2	Location 3	Location 4	Clues
1	B	G	Y	Y	WH
2	Y	Y	G	B	WH
3	G	B	B	Y	WH
CODE	R G Y B	R G Y B	R G Y B	R G Y B	

E) **(L)**

Guess	Location 1	Location 2	Location 3	Location 4	Clues
1	Y	Y	G	B	WH-WH-WH
2	G	B	B	Y	BL-BL
3	B	B	R	R	BL-WH
CODE	R G Y B	R G Y B	R G Y B	R G Y B	

EXPLORE (4)! In the 4 location game (standard), with one guess, can you ever get theses clues:

	Clue	Result is…	If impossible, why?
A)	WH – WH – WH – BL	Possible Impossible	
B)	BL – BL – BL – WH	Possible Impossible	
C)	BL – BL – BL – BL	Possible Impossible	
D)	BL – WH	Possible Impossible	

For Love of the Math: *A mathematican named Donald Knuth came up with a method to solving the Mastermind game. His article is here: http://www.dcc.fc.up.pt/~sssousa/RM09101.pdf*
An extension of his theories can win the game in less than 5 moves on average! Here's a simplified version that can guarantee you the code broken in 6 moves (7th required to guess):
(R, O, O, R), (O, Y, B, G), (Y, Y, R, R), (G, B, O, G), (B, P, B, P), (P, P, G, Y)

<u>2.10: Sets – The Basis for Counting</u>

In mathematics, we often deal with collections of objects called **<u>sets</u>**. To represent these, we often use set notation which uses '{' and '}' around the objects, with commas separating the objects. Some of our set rules are listed here:

- An object is either in the set or not in the set.
- The order of objects in the set doesn't matter: {2, 5} is the same set as {5, 2}
- Repeating objects in the set doesn't change the set, so we typically don't do it. {3, 3, 3, 5} is the same set as {3, 5} because both sets have a 3 and a 5.

A set can be represented completely (called list or roster notation), or given as more of a description. Imagine trying to list the set of states in the United States of America… it's just long.

$S = \{x \mid x \text{ is a state in the USA}\}$ is the notation that expresses the same idea. We read this as: "Set *S* is the set of all objects *x* <u>*as long as*</u> *x* is a state in the USA." Sometimes this is shortened to "Set *S* is the set of all states in the USA."

The objects in a set are called **<u>elements</u>**, and the symbol $\in$ is the symbol we use related to elements. $7 \in \{3, 5, 7\}$ means that 7 is an element of the set containing 3, 5, and 7. $4 \notin \{3, 5, 7\}$ means that 4 is not an element of the set containing 3, 5, and 7.

<u>CAUTION!</u> There are often mistakes made between sets and elements. Here's a few examples and counterexamples showing those areas. One way to see the set brackets is like a bag. The objects inside the bag are elements. However, it is possible to have a bag inside of a bag – IN-SET-TION. See, it rhymes with Inception (movie about dreams within dreams).

<u>Examples:</u> Determine if the object is an element of a set or not.

A) $4 \in \{3, 4, 5, 7\}$

B) $4 \in \{3, \{4\}, 5, 7\}$

C) $\{4\} \in \{3, \{4\}, 5, 7\}$

D) $\{4\} \in \{3, 4, 5, 7\}$

<u>Solutions:</u>

A) Think of $\{3, 4, 5, 7\}$ as a bag that has the numbers 3, 4, 5, and 7 inside. The statement $4 \in \{3, 4, 5, 7\}$ is asking whether 4 is one of the numbers in the bag. Since it is, this is true!

B) Think of $\{3, \{4\}, 5, 7\}$ as a bag that has the numbers 3, 5, and 7 inside, as well as another bag with a number 4. So when you open it up the outer bag, you see three numbers and a bag. The statement $4 \in \{3, \{4\}, 5, 7\}$ is asking whether 4 is one of the numbers in the outer bag. Since 4 is not a number in the outer bag, this is false!

C) Think of $\{3, \{4\}, 5, 7\}$ as a bag that has the numbers 3, 5, and 7 inside, as well as another bag with a number 4. So when you open it up the outer bag, you see three numbers and a bag. The statement $4 \in \{3, \{4\}, 5, 7\}$ is asking whether there is a bag containing a 4, written as $\{4\}$, in the outer bag. Since $\{4\}$ is an object in the outer bag, this is true!

D) Think of $\{3, 4, 5, 7\}$ as a bag that has the numbers 3, 4, 5, and 7 inside. The statement $\{4\} \in \{3, 4, 5, 7\}$ is asking whether there is a bag containing a 4, written as $\{4\}$, in the outer bag. Since $\{4\}$ is not an object in the outer bag, this is false!

Okay, now I'll ask about the set of all good disco bands… how many are there? None. Exactly. This is an example of a set with no elements, which we can represent like $\{\ \}$ or as $\varnothing$ and we call it the **empty set**. Some of you may disagree, and you may believe that there are good disco bands. So for you, it may be better to consider the set of all negative numbers greater than zero. How many are there? None. Exactly. That may be a better, but less fun, example of an empty set.

EXPLORE (1)! Determine whether these statements are true for $A = \{1, 6, \Delta\}$ and $B = \{2, 6, 23, \Delta, A\}$.

	Statement	Truth value
A) **	$6 \in A$	True False
C) **(R)**	$8 \in A$	True False
E) **(R)**	$\Delta \notin A$	True False
G) **(R)**	$8 \in \varnothing$	True False
I) **(R)**	$8 \notin \varnothing$	True False
K) **(R)**	$\{6\} \in A$	True False

	Statement	Truth value
B) **	$6 \in B$	True False
D) **(L)**	$8 \notin B$	True False
F) **(L)**	$A \in B$	True False
H) **(L)**	$B \in \varnothing$	True False
J) **(L)**	$23 \notin B$	True False
L) **(L)**	$\{6\} \notin B$	True False

At this point we have objects called sets and objects called elements. It is possible that some objects are in more than one set, like the 6 was in the previous exploration. It is also possible that every element of one set is contained in another. Picture sets $D = \{1, 6, \Delta\}$ and $E = \{1, 6, 23, \Delta, 57\}$. If you were comparing the two sets, you would say that every element of *D* is contained in *E*. This relationship is called a **subset**, and the symbol $\subseteq$ is the symbol we use related to subsets. $D \subseteq E$ means that every element in *D* is an element of *E*, so *D* is a subset of *E*. $E \not\subseteq D$ means that *E* is not a subset of *D*, which means that there is at least one element of *E* that is not in *D*. Remember that this symbol relates to sets (not elements).

EXPLORE (2)! Determine whether these statements are true for $A = \{1\}$ and $B = \{1, 6, 23, \Delta, A\}$.

	Statement	Truth value
A) **	$6 \subseteq B$	True False
C) **(L)**	$\{6, 23\} \subseteq B$	True False
E) **(L)**	$\Delta \subseteq A$	True False
G) **(L)**	$1 \subseteq B$	True False

	Statement	Truth value
B) **	$\{6\} \subseteq B$	True False
D) **(R)**	$B \subseteq \{6, 23\}$	True False
F) **(R)**	$A \subseteq B$	True False
H) **(R)**	$\{1\} \subseteq B$	True False

The definition of subset shows a nice connection between logic and subsets:

- $D \subseteq E$ *if and only if* all elements of *D* are elements of *E*.

The quantified statement "All elements of D are elements of E" means the same thing as this conditional statement: "if $x \in D$, then $x \in E$." To prove a conditional statement false, we need just one counter-example… just one time when the condition is met but the conclusion fails. From the previous exploration, there was the part saying $B \subseteq \{6, 23\}$. Is this true or false? We said false, but why? 1 is an element of B but 1 is not an element of $\{6, 23\}$, so $B \subseteq \{6, 23\}$ is false. [Only need one counter-example; there are many.]

<u>Example:</u> Determine the truth value of: $\varnothing \subseteq \{6, 23\}$

In order to prove this false, we need one counter-example. We need to find an element of $\{6, 23\}$ that is not in $\varnothing$. Well, that's a pretty good problem since $\varnothing$ is the empty set and has no elements. There is no way we can prove $\varnothing \subseteq \{6, 23\}$ false, so it must be true!

<u>**EXPLORE (3)!**</u> Determine whether these statements are true for $A = \{1\}$, $B = \{1, 6, 23, \Delta, A\}$, and $C = \{1, \varnothing\}$.

	Statement	Truth value		Statement	Truth value
A) **	$B \subseteq \varnothing$	True False	B) **	$\varnothing \subseteq \{6\}$	True False
C) **	$\varnothing \subseteq \varnothing$	True False	D) **	$\varnothing \in \varnothing$	True False
E) **(L)**	$\Delta \notin \varnothing$	True False	F) **(R)**	$\varnothing \subseteq A$	True False
G) **(L)**	$\varnothing \in A$	True False	H) **(R)**	$\varnothing \in B$	True False
I) **(L)**	$6 \in \varnothing$	True False	J) **(R)**	$A \notin \varnothing$	True False
K) **(L)**	$\varnothing \in C$	True False	L) **(R)**	$\varnothing \subseteq C$	True False

<u>Interactive Example:</u> Now that you have an idea about what it means to be a subset, like $A \subseteq B$, what would happen if both $A \subseteq B$ and $B \subseteq A$?

A) $A \subseteq B$ in words means…

B) $B \subseteq A$ in words means…

C) What must be true about these two sets?

Any time that two sets have exactly the same elements, we say that they are <u>**equal sets**</u> and use the traditional symbol =. This new concept again links back to logic with the definition:

- $A = B$ <u>*if and only if*</u> $A \subseteq B$ and $B \subseteq A$.

EXPLORE (4)! Determine whether these statements are true for $A = \{1, 6\}$, $B = \{1, 6, 23, \Delta, A\}$, and $C = \{6, 1\}$.

	Statement	Truth value
A) **	$A \subseteq C$	True False
C)	$A = C$	True False
E)	$B \subseteq C$	True False

	Statement	Truth value
B)	$C \subseteq A$	True False
D)	$C \subseteq B$	True False
F)	$B = C$	True False

For every set of objects we can think about, we could also think about all the things that aren't in that set. So for set *B*, the set of all things not in *B* would be called the **complement** and written as $\overline{B}$. This new set has everything not in set B, which could include things that don't make much sense in the context of a problem, so mathematicians typically restrict the possibilities for elements by defining a **universe**. The universe simply gives the description of all possible elements for a given problem. The universe is typically represented as *U*.

EXPLORE (5)! Determine whether these statements are true for $U = \{1, 2, 3, 4, 5, 6\}$ $A = \{1, 6\}$, and $B = \{1, 2, 3\}$.

	Statement	Truth value
A) **	$5 \in \overline{B}$	True False
C)	$6 \in A$	True False
E)	$5 \in \overline{U}$	True False

	Statement	Truth value
B)	$3 \in \overline{B}$	True False
D)	$3 \in \overline{A}$	True False
F)	$\overline{U} = \varnothing$	True False

The last portion of this section involves counting how many elements are in a set. We call this the **cardinal number** of a set and use the notation $n(A)$.

EXPLORE (6)! Determine the following values for $U = \{1, 2, 3, 4, 5, 6\}$ $A = \{1, 6\}$, and $B = \{1, 2, 3\}$.

A) $n(U)$

C) **(L)** $n(A)$

E) **(L)** $n(\overline{B})$

B) **(R)** $n(B)$

D) **(R)** $n(\overline{A})$

F) $n(\overline{U})$

<u>EXPLORE (7)!</u> Try a few more involving cardinality.

A) ** M = {x | x is a month of the year}. $n(M)$ =

B) **(L)** S = {x | x is a state in the United States}. $n(S)$ =

C) **(R)** F = {x | x is a senator in the United States senate}. $n(F)$ =

D) **(RL)** If $n(J) = 0$, what does that tell you about set J?

E) **(L)** If $n(P) = n(Q)$, can we conclude that $P = Q$? Explain.

F) **(R)** If $P = Q$, can we conclude that $n(P) = n(Q)$? Explain.

G) **(L)** If $n(P) \leq n(Q)$, can we conclude that $P \subseteq Q$? Explain.

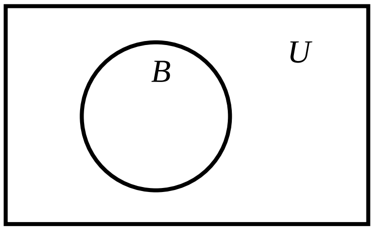
H) **(R)** If $P \subseteq Q$, can we conclude that $n(P) \leq n(Q)$? Explain.

2.11: Sets – Basic Operators and Venn Diagrams

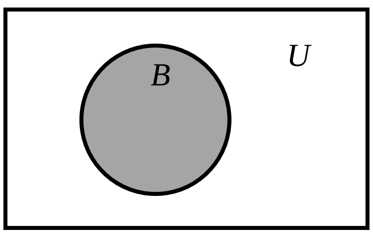
Now that we have a base knowledge of sets, we can start thinking about set operations. In this section, each of these set concepts will be represented visually with a Venn diagram. The first is to think about a set complement from the previous section. $\overline{B}$ is the set of all elements that are not in *B*.

Example: Create two Venn diagrams – one shading set *B* and the other shading $\overline{B}$.

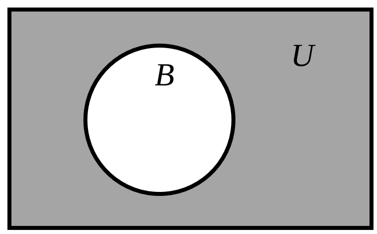
Start with a Venn diagram labeling the set *B* and the universe *U*, which is everything inside the box.



Then shade the two sets appropriately.



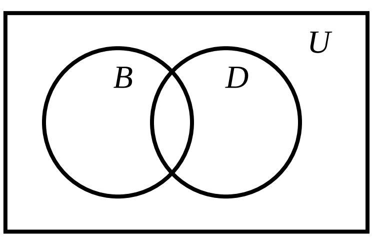
Set *B*



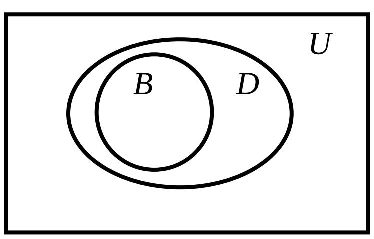
Set $\overline{B}$

EXPLORE (1)! Describe the region in words and then shade the appropriate regions in the graph.

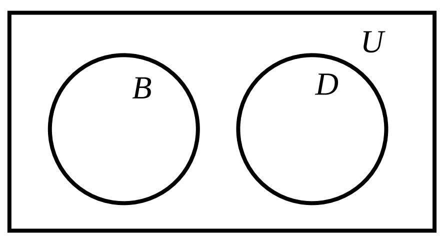
A) ** $\overline{B}$



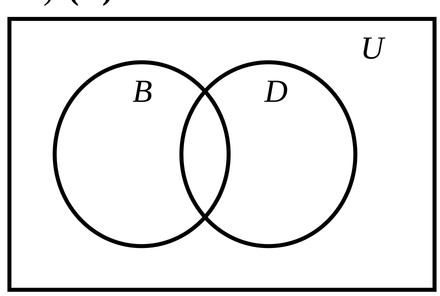
B) $\overline{B}$



C) **(L)** $\overline{D}$



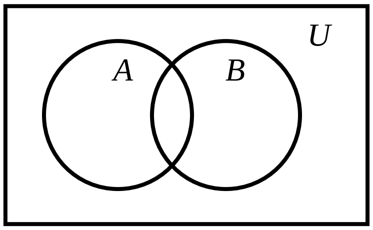
D) **(R)** $\overline{U}$



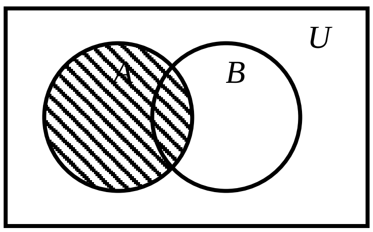
Here's one new piece that we haven't dealt with yet, $A \cap B$, and is read "*A* intersect *B*." This symbol represents set **<u>intersection</u>**, and here's the definition: $x \in (A \cap B)$ if and only if $x \in A$ and $x \in B$. In order to be in the intersection, an element must be in both sets.

<u>Example:</u> Shade set $A \cap B$.

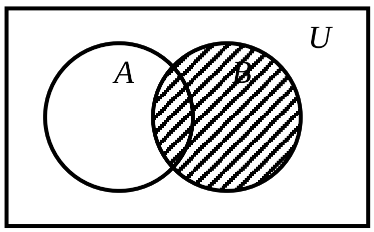
Start with a Venn diagram labeling the sets *A* and *B* and the universe *U*, which is everything inside the box.



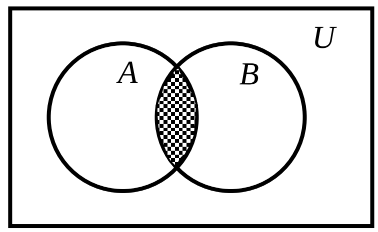
Then shade set *A* and set *B* separately. Finally, shade the intersection which is the region shaded in <u>*both*</u> diagrams.



Set *A*



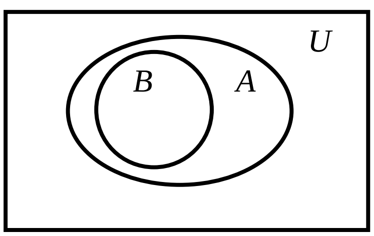
Set *B*



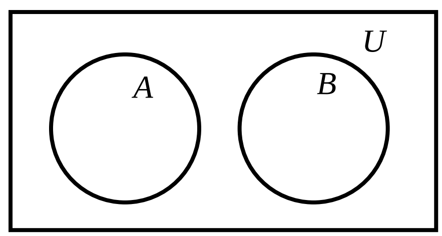
Set $A \cap B$

<u>EXPLORE (2)!</u> Describe in words and then shade the appropriate regions in the graph.

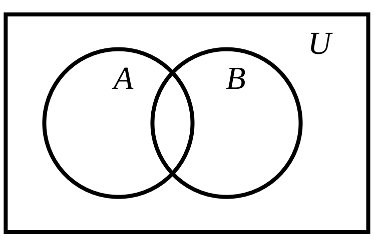
A) ** $A \cap B$



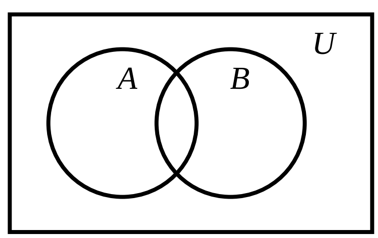
C) **(L)** $A \cap B$



B) $B \cap \overline{A}$



D) **(R)** $A \cap \overline{B}$

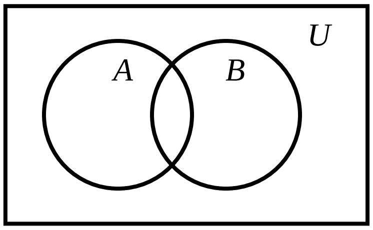


In part (C), when sets don't overlap/intersect, we call them **<u>disjoint</u>** sets (or **<u>mutually exclusive</u>** sets) and the sets have nothing in common.

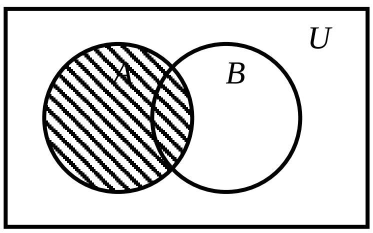
Another set piece that is new is $A \cup B$, which is read "*A* union *B*." This symbol represents set **<u>union</u>**, and here's the definition: $x \in (A \cup B)$ if and only if $x \in A$ or $x \in B$. In order to be in the intersection, an element must be in at least one of the sets and could be in both.

<u>Example:</u> Shade set $A \cup B$.

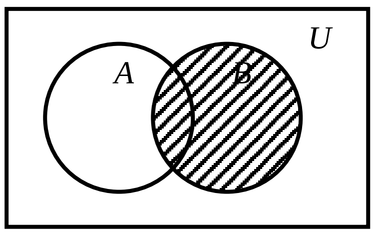
Start with a Venn diagram labeling the sets *A* and *B* and the universe *U*, which is everything inside the box.



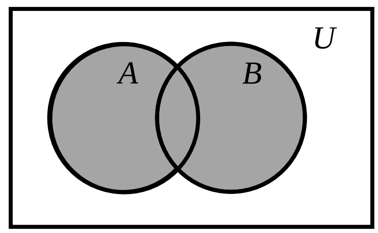
Then shade set *A* and set *B* separately. Finally, shade the union which is the region shaded in <u>*either or both*</u> of the diagrams.



Set *A*



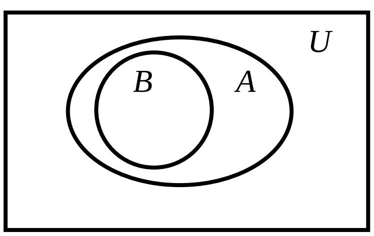
Set *B*



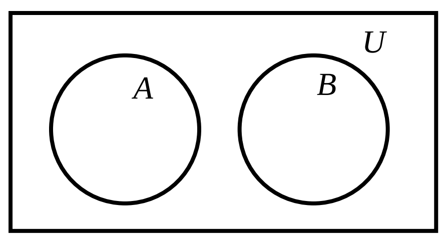
Set $A \cup B$

<u>EXPLORE (3)!</u> Describe in words then shade the appropriate regions in the graph.

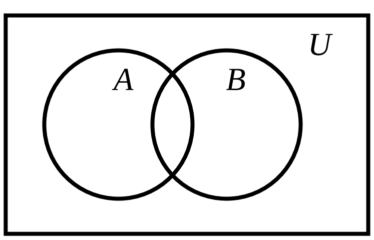
A) ** $A \cup B$



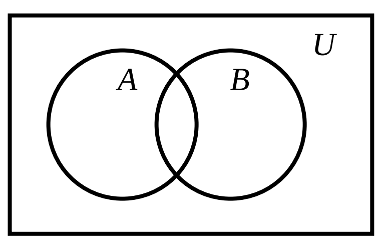
C) **(L)** $A \cup B$



B) $B \cup \overline{A}$



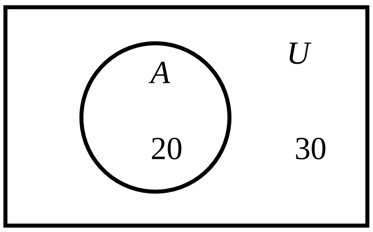
D) **(R)** $A \cup \overline{B}$



Now that we have the idea, let's put it together and link it back to the standard math operations. In order to see this connection, we'll start with some basic problems about sets.

<u>Example:</u> If $n(U)=50$ and $n(A)=20$, what is $n(\overline{A})$? Fill out a Venn diagram to show the values.

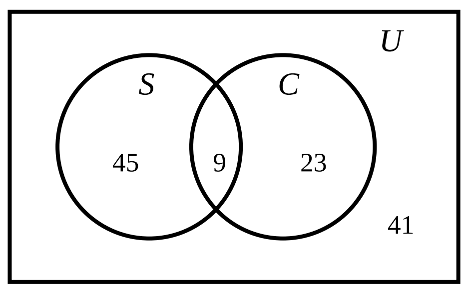
Since $n(U)=50$, there are 50 objects total in the entire box. However, there are two regions here. $n(A)=20$ tells us that there are 20 objects inside of *A*, so we could write 20 there. Since the box has 50 total objects, there must be 30 objects outside of *A*. So $n(\overline{A})=30$ and we can write 30 outside of *A*. The Venn diagram below shows all of the information.



Based on the picture and the example above, what must be true about $n(A)+n(\overline{A})$?

$n(A)+n(\overline{A})$ = ______________ because…

<u>EXPLORE (4)!</u> Interpret the Venn diagram here relating people who wear shorts (S) and those who drink coffee (C).



A) ** Write in words then find $n(S)$.

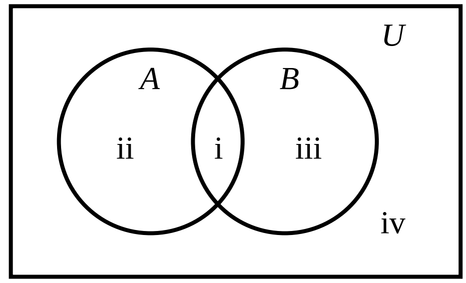
B) **(L)** Write in words then find $n(C)$.

C) **(R)** Write in words then find $n(S \cap C)$.

D) Write in words then find $n(S \cup C)$.

E) Write in words then find $n(\overline{C})$.

<u>EXPLORE (5)!</u> There are 4 distinct regions in the Venn diagram.



A) **(L)** A is made up of what regions?

B) **(R)** B is made up of what regions?

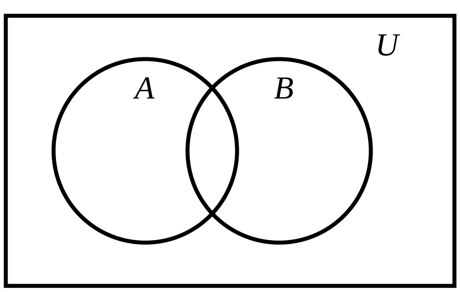
C) **(L)** $A \cup B$ is made up of what regions?

D) **(R)** $A \cap B$ is made up of what regions?

E) Is $n(A \cup B) = n(A) + n(B)$ always true? Look at the regions making up each and explain your answer.

F) Create an equation that fixes $n(A \cup B) = n(A) + n(B)$ into an equation that is always true.

<u>Interactive Example:</u> If $n(U) = 50$, $n(A) = 20$, $n(A \cap B) = 8$, and $n(B) = 12$, fill out the rest of the numbers in the Venn diagram. **<u>NOTE:</u>** Since we aren't sure whether this is an all, some, or none type of Venn diagram, you should write it as "some" because that type can work for all three types.



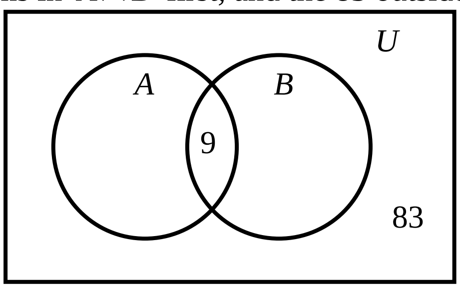
Because both *A* and *B* include $A \cap B$, adding $n(A)+n(B)$ will actually count the intersection twice. The only time $n(A \cup B)=n(A)+n(B)$ will be true is if the intersection is empty!

In order for us to end up with a correct equation all the time, we need to remove one of the intersections. The correct equation representing the relationship is: $n(A \cup B)=n(A)+n(B)-n(A \cap B)$.

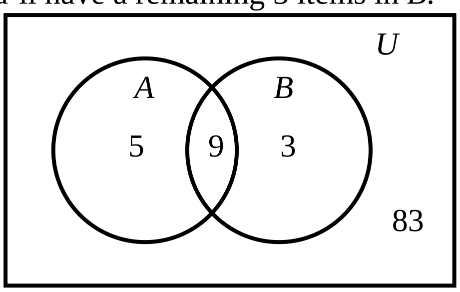
<u>Example:</u> If $n(U)=100$, $n(A)=14$, $n(A \cap B)=9$, and $n(B)=12$, find $n(A \cup B)$ and then fill out the rest of the numbers in the Venn diagram.

Using the equation we just came up with, $n(A \cup B)=n(A)+n(B)-n(A \cap B)$, plug in the pieces we know.

$n(A \cup B)=n(A)+n(B)-n(A \cap B) \Rightarrow n(A \cup B)=14+12-9=26-9=17$. So there are 17 items in $A \cup B$, and with 100 items in the universe, there must be 100 – 17 = 83 items outside of $A \cup B$. Write in the 9 items in $A \cap B$ first, and the 83 outside of $A \cup B$.



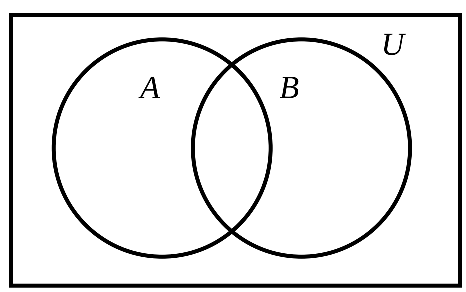
To find what remains for the Venn diagram, remember that $n(A)=14$. But we already used up 9 of those items that were also in *B*, so there must be 14 – 9 = 5 items left over in *A*. Repeat that for $n(B)=12$ and you'll have a remaining 3 items in *B*.



Now check our work – does everything match? Is each one of these things true based on our picture?

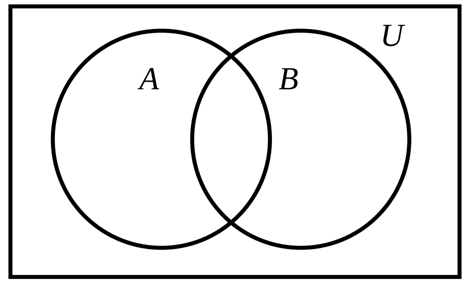
- $n(U)=100$
- $n(A)=14$
- $n(A \cap B)=9$
- $n(B)=12$

<u>EXPLORE (6)!</u> If $n(U)=483$, $n(A)=147$, $n(A \cap B)=29$, and $n(B)=204$, find $n(A \cup B)$ and then fill out the rest of the numbers in the Venn diagram.

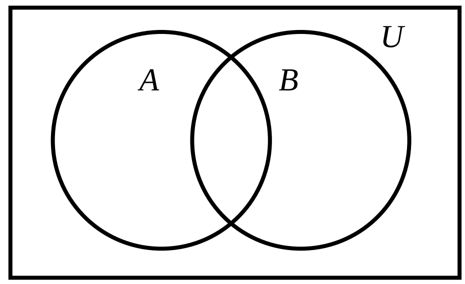


EXPLORE (7)! Find $n(A \cup B)$ and then fill out the rest of the numbers in the Venn diagram.

A) ** $n(U) = 66$, $n(A) = 8$, $n(A \cap B) = 0$, and $n(B) = 11$.

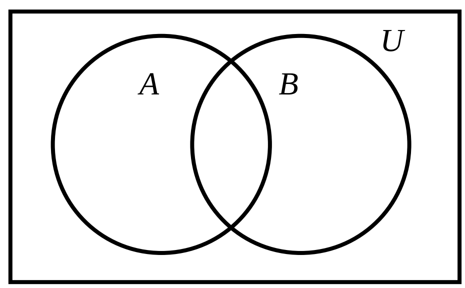


B) $n(U) = 300$, $n(A) = 121$, $n(A \cap B) = 1$, and $n(B) = 101$.

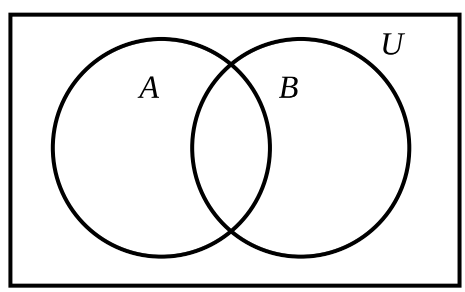


EXPLORE (8)! Find $n(A \cap B)$ and then fill out the rest of the numbers in the Venn diagram.

A) ** $n(U) = 550$, $n(A) = 400$, $n(A \cup B) = 450$, and $n(B) = 300$.



B) $n(U) = 975$, $n(A) = 284$, $n(A \cup B) = 800$, and $n(B) = 516$.



EXPLORE (9)! At times, especially in statistics or real life, there may not be enough information to find all the values. Here are 8 situations, but only 4 of them have enough information to complete the Venn diagrams. If a situation has enough information, use the information to complete a Venn diagram at the bottom of the page – be sure to label the Venn diagram with which part you're referring to. If a situation doesn't have enough information, explain why.

A) $n(U)=66$, $n(A)=8$, and $n(A\cap B)=0$.

B) $n(U)=66$, $n(A)=8$, $n(A\cap B)=8$, and $n(B)=11$.

C) $n(A)=8$, $n(A\cap B)=7$, and $n(B)=40$.

D) $n(U)=66$ and $n(A\cup B)=0$

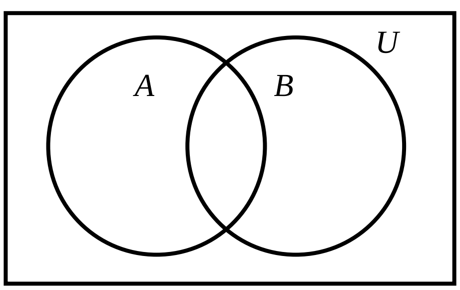
E) $n(U)=66$, $n(A)=8$, and $n(A\cup B)=8$.

F) $n(U)=66$, $n(A)=0$, and $n(A\cup B)=8$.

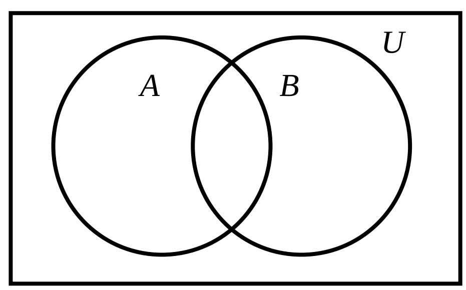
G) $n(U)=66$, $n(A)=8$, $n(A\cup B)=18$, and $n(B)=11$.

H) $n(U)=66$, $n(A\cup B)=38$, and $n(A\cap B)=15$.

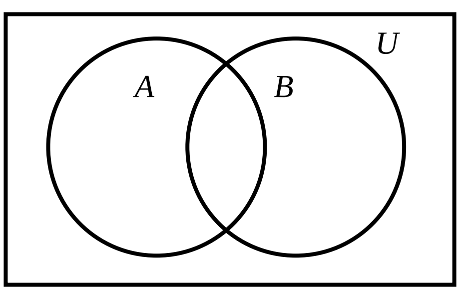
Part ________



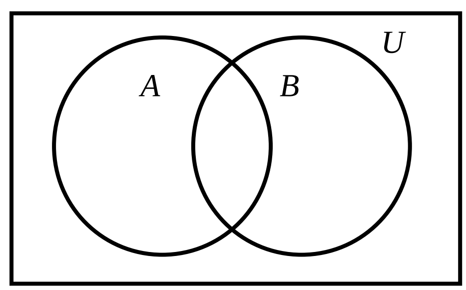
Part ________



Part ________



Part ________



2.12: Using Sets to Solve Problems

As you prepare for statistics courses, one of the ways to gather data is a survey. We often get phone calls asking for our help in a survey or at the mall when someone with a clipboard follows you around. We will deal with basic surveys here that have only a few questions.

Example (1): 200 surveys are given out to students at MiraCosta asking what classes students are taking. There are 2 boxes: ☐ Math ☐ English. When collected, the following information is obtained.

- 19 are blank.
- 128 have Math checked.
- 150 have English checked.

A) **(L)** Would you be able to say that *all* people who checked English also checked math? Explain.

B) **(R)** Would you be able to say that *all* people who checked math also checked English? Explain.

C) Would you be able to say that *some* people who checked math also checked English? Explain.

D) What type of Venn diagram should you draw for this problem: All, Some, or None? Explain.

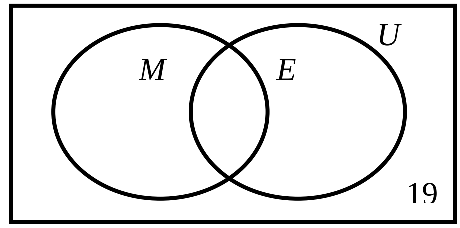
Determine the number of students taking both Math and English, and what percentage this is out of the total number of surveys.

Example (2): 200 surveys are given out to students at MiraCosta asking what classes students are taking. There are 2 boxes: ☐ Math ☐ English. When collected, the following information is obtained.

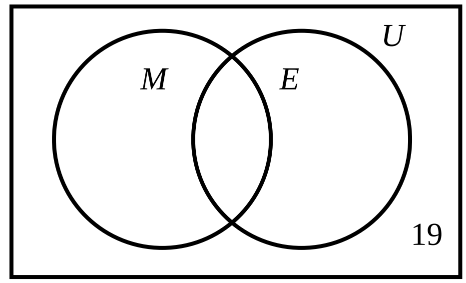
- 19 are blank.
- 128 have Math checked.
- 150 have English checked.

Determine the number of students taking both Math and English, and what percentage this is out of the total number of surveys.

Solution: Drawing a Venn diagram can help us organize our thoughts. We'll call Math (M) and English (E) to make it easy to see. Then we'll put in what we know. Now that the Venn diagram is labeled, we can put in the 19 blank surveys. These would be located outside of both circles.



With 200 surveys passed out, and only 19 used so far, there are 200 – 19 = 181 remaining that would be in *M* or *E* or both. Thankfully, *M* or *E* or both is $M \cup E$.



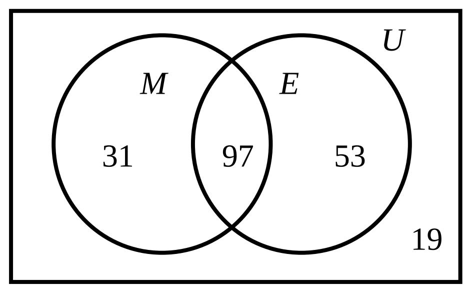
So we know that $n(M \cup E) = 181$. From the problem, we also know $n(M) = 128$ and $n(E) = 150$. We can use the formula: $n(M \cup E) = n(M) + n(E) - n(M \cap E) \Rightarrow$
$181 = 128 + 150 - n(M \cap E) \Rightarrow$
$181 = 278 - n(M \cap E) \Rightarrow$
$n(M \cap E) = 278 - 181 = 97$

So there are 97 students who are taking both Math and English. Now we can fill out the rest of the diagram.

For the math circle: 128 – 97 = 31 (not in both)

For the English circle: 150 – 97 = 53 (not in both)

The 97 students out of 200 is a percentage of $\frac{97}{200} = 0.485 = 48.5\%$.



EXPLORE (1)! 250 surveys are given out to people walking through the mall. There are 2 boxes to choose from: ☐ PS4 ☐ Xbox One. When collected, the following information is obtained.

- 111 are blank.
- 103 have PS4 checked.
- 57 have Xbox One checked.

A) Would you be able to say that ***all*** people who checked Xbox One also checked PS4? Explain.

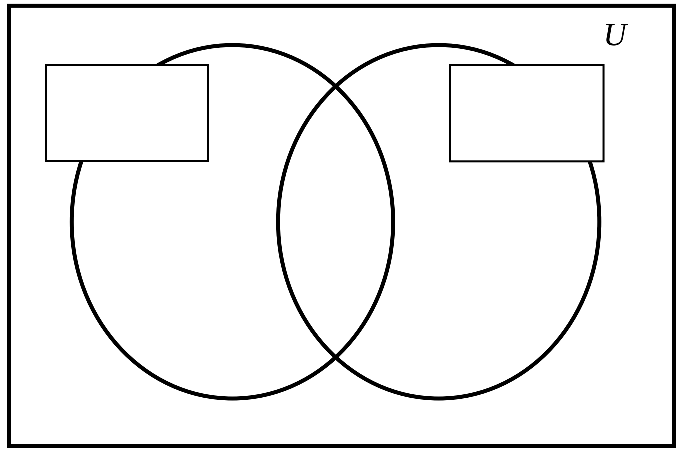
B) Would you be able to say that *some* people who checked Xbox One also checked PS4? Explain.

C) What type of Venn diagram should you draw for this problem: All, Some, or None? Explain.

EXPLORE (2)! 250 surveys are given out to people walking through the mall. There are 2 boxes to choose from: ☐ PS4 ☐ Xbox One. When collected, the following information is obtained.

- 111 are blank.
- 103 have PS4 checked.
- 57 have Xbox One checked.

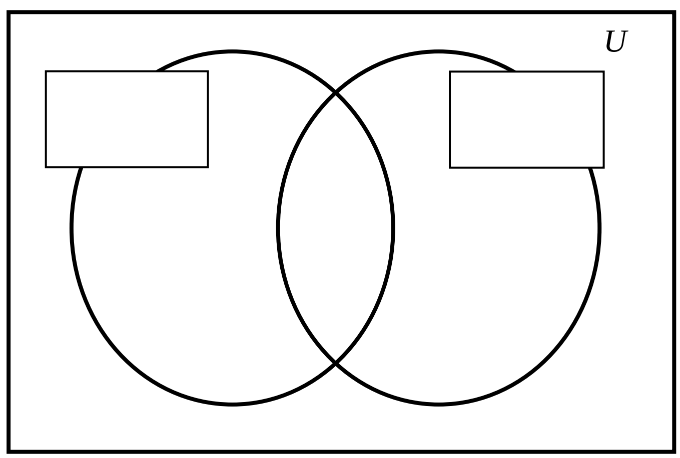
Determine the number of people who own both a PS4 and an Xbox One, and what percentage this is out of the total number of surveys.



EXPLORE (3)! 587 people are surveyed about their cell phone provider and given two questions: (1) "Are you happy with the service?" and (2) "Are you with Sprint?" The following information is obtained.

- 211 said they are with Sprint.
- 421 said they were happy with the service.
- 139 said they are not happy and not with Sprint.

Determine the number of people who are happy with Sprint, and what percentage this is out of the total number of surveys.



2.13: Basic Counting Techniques

We can see how to count a lot with Venn diagrams, and sets are the basis that we start with for counting.

Let's start with thinking of making outfits. As a math teacher, there are polo shirts and shorts. Imagine a math teacher who has 4 different pairs of shorts and 7 polo shirts.

Example: How many different outfits can be made for the math teacher above?

If we create sets for these objects, they would look like this:

- A set of 7 polo shirts is $P = \{PS_1, PS_2, PS_3, PS_4, PS_5, PS_6, PS_7\}$
- A set of 4 pairs of shorts is $S = \{S_1, S_2, S_3, S_4\}$

To make an outfit, we need to select one polo shirt and one pair of shorts. If we pick the first polo shirt, there are 4 choices for shorts to go along with it. The same is true for each of the 7 polo shirts. So the number of outfits is: $4+4+4+4+4+4+4 = 7 \times 4 = 28$.

EXPLORE (1)! Determine the number of options based on the information given.

A) ** How many different outfits can be made for a math teacher who has 8 pairs of slacks and 12 blouses and 6 pairs of shoes?

B) **(L)** How many ways can you eat a dinner option if you have to pick a salad, a main entrée, and a dessert… and there are 4 salad options, 8 main entrées, and 3 dessert choices?

C) **(R)** A noodle company serves 14 types of noodles, 6 protein options, and 11 sauces. If you choose a different order every day, how many days can you eat before you repeat a dish?

The concept illustrated above is called the **Fundamental Counting Principle**, and it says that if you have *m* choices for the first option and *n* choices for the second, then there are $m \times n$ ways to pick one of each. This can be extended to as many options as you want.

We could even use this principle if you wanted to see how many ways there are to select something, replace it in the group, and then select again. Sometimes card games are like this.

Example: You're going to draw two cards from a deck – you draw one, write it down and put it back in the deck, then shuffle and draw again. How many different two-card hands could you get?

There are 52 cards in the deck. Each option has 52 choices, so there are 52×52 = 2,704 two-card hands. Sometimes, with sets *A* and *B*, $n(A) = n(B)$. And when the cardinal number is the same, we can pair up the elements in a **one-to-one correspondence**. In a swim meet with 4 lanes and 4 swimmers, this can give us a way to figure out how many different line-ups are possible.

Example: Determine how many line-ups are possible with 4 lanes and 4 swimmers.

We can start with the first swimmer and see about how many lanes are an option? (4)
Now move to the next swimmer, but now there are fewer lanes – how many now? (3)
For the next swimmer, how many lanes are left to choose from? (2)
And finally, the last swimmer has how many lanes to choose from? (1)

So based on the Fundamental Counting Principle, the total number of line-ups is $4 \times 3 \times 2 \times 1 = 24$.

EXPLORE (2)! Determine the number of ways to do what is described.

A) ** Determine how many different orders are possible for a beauty pageant with 5 time-slots and 5 contestants.

B) Determine how many ways we can pair up 7 Bridezillas with 7 amazing gowns?

Each time, we seem to be multiplying numbers in decreasing order to form these 1-to-1 correspondences. When we see these things in a pattern, we can create a new method to calculate these results more quickly. Mathematically, we call this a **factorial** and the symbol we use is '!' as in 4!.

- $4 \times 3 \times 2 \times 1 = 4!$
- $5 \times 4 \times 3 \times 2 \times 1 = 5!$
- $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6!$
- $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 7!$

Your calculator has this built in, and if using the key marked with PRB, and then look for the ! symbol (if using the TI-30XIIS), or shift x^{-1} on the Casio fx-300ES Plus.

<u>EXPLORE (3)!</u> Practice using the calculator on these and find the value.

A) ** 7!

C) **(L)** 6!

B) 5!

D) **(R)** 4!

Let's figure out how good our calculator is.

<u>EXPLORE (4)!</u> Using the factorial feature on your calculator, discover the following (working with a few others would be great and save time).

A) When does the calculator move into scientific notation?

a. This number shows the full value: ________

b. This number shows the scientific notation: __________

B) When does the calculator stop showing any value?

a. This number shows the scientific notation: __________

b. This number breaks the factorial feature of the calculator: __________

C) Why does the calculator "break"?

We can think of arranging objects in order as a type of 1-to-1 correspondence. If there were 5 objects to put in order, there are 5 choices for the first object, 4 for the next, 3 for the next, and so on. So the total number of ways to arrange 5 objects in order is 5!.

<u>EXPLORE (5)!</u> Determine the number of ways that the objects can be arranged *<u>in order</u>*.

A) ** There are 5 different books on a shelf.

B) **(L)** There are 12 different shirts on hangers in a closet.

C) **(R)** There are 8 students in a spelling bee.

D) There are 0 healthy choices on the menu.

This last example is very interesting because it really represents 0!, so let's make sure that the calculator has this correct. How many ways can you choose from 0 items? Well, you just don't pick any of them which is the only choice. So 0! should be 1. Is that what your calculator shows?

Of course, what happens if we don't select all of the objects (or people)?

Example: How many ways can we arrange 3 books on the shelf out of 6 different books?

There are 6 books for the first choice, 5 books for the second choice, and 4 books for the last choice. So the number of ways to arrange the 3 books is $6\times5\times4$ = 120.

When we have 6 different books but are selecting only 3 of them in order, then we saw that there are 120 choices coming from $6\times5\times4$. This sure seems like factorial, but because it doesn't go all the way down to 1, we can't use 6!. 6! would be used when we are arranging all 6 different books in order.

$6!=6\times5\times4\times3\times2\times1$, but this is too large. From fractions and FLOF, we know that we can remove common factors. Let's connect factorial (choosing all the options) with this new idea (choosing only some of them).

$$6\times5\times4=\frac{6\times5\times4\times3\times2\times1}{3\times2\times1}=\frac{6!}{3!}$$

This new option is used so much that it is programmed into your calculators and called **permutations**. With permutations, you are selecting a certain number of objects, *r*, from a group of *n* different objects, and you can't replace the objects. The calculator symbol is ${}_nP_r$ and is under the PRB menu as well on the TI-30XIIS, or shift × on the Casio fx-300ES Plus. The formula is ${}_nP_r=\dfrac{n!}{(n-r)!}$

EXPLORE (6)! Use your calculator to determine the following values:

A) ** ${}_7P_4$

B) ${}_{11}P_4$

C) **(L)** ${}_{14}P_{12}$

D) **(R)** ${}_{18}P_{17}$

E) **(L)** ${}_{97}P_4$

F) **(R)** ${}_{10}P_3$

The calculator actually gives the opportunity to solve problems that would not be possible with just factorial. Consider ${}_{97}P_4$ and type it into your calculator as factorials: ${}_{97}P_4=\dfrac{97!}{(97-4)!}=\dfrac{97!}{93!}$. Does the calculator work with 97!? What about 93!? Does the calculator work with ${}_{97}P_4$?

Interactive Example: Find the results by writing it out in the formula then checking with the calculator. Interpret what each means.

A) ** ${}_7P_0$

C) **(L)** ${}_7P_2$

B) **(R)** ${}_7P_7$

D) 7!

EXPLORE (7)! Put these in your own words:

A) Why is ${}_7P_7$ the same as 7!?

B) Why is ${}_7P_0$ the same as 1?

Now that we understand more about permutations, we can put these into practice.

EXPLORE (8)! Determine the number of ways to select the following.

A) ** Ranking the top 5 books, in order, out of a book of the month club (12 different books).

B) **(L)** Ranking the top 4 contestants, in order, out of a group of 51 competing for Miss Teen USA.

C) **(L)** Ranking the top 7 students, in order, in a scholarship competition with 100 different students.

D) **(R)** Ranking the top 3 burritos, in order, in a taste-test challenge with 400 burritos.

E) **(R)** Ranking the top 8 t-shirts, in order, from your closet that has 36 t-shirts in it.

2.14: Advanced Counting Techniques

Permutations are pretty cool, and they work well when all the objects are different.

Example 1: How many ways can we rewrite the word STOP?

There are 4 letters, and rewriting all 4 in order would be the same as ${}_4P_4 = \frac{4!}{(4-4)!} = \frac{4!}{0!} = 4! = 24$.

Example 2: If the letters S-T-O-P are put into a bag and you draw a letter, then write it down and put it back before you draw again, how many 4-letter options are possible?

Since we are replacing the letters after we record it, we can't use factorial and can't use permutations. We must look at the Fundamental Counting Principal and find that there are 4 choices at first, then 4 for the next, and really 4 each time we draw. So the number of options is $4 \times 4 \times 4 \times 4 = 4^4 = 256$.

EXPLORE (1)! Determine the number of ways to arrange the objects listed:

A) ** How many ways can you rank the top 4 bagel types, in order, from a set of 13 bagel types?

B) There are 6 different colored pegs and you are making a 4 color code where you can repeat options. How many "codes" can be made? (This is the number of different MasterMind codes).

C) How many ways are there to rearrange the letters in STRAYING?

With different words, the number of ways to rearrange them might be different if some of the letters are the same. Think of the word ART. There are 3! ways to rearrange these letters, or ${}_3P_3 = 6$. To show all of these options, we could list them all out:

- ART
- ATR
- RAT
- RTA
- TAR
- TRA

Again, we could do this for STOP as done earlier on this page. Here's all 24 ways:

- STOP
- STPO
- SOTP
- SOPT
- SPOT
- SPTO
- TOPS
- TOSP
- TPSO
- TPOS
- TSPO
- TSOP
- OPTS
- OPST
- OSTP
- OSPT
- OTPS
- OTSP
- POTS
- POST
- PTSO
- PTOS
- PSTO
- PSOT

Example: Look at the word ZOO; how many ways could we rearrange this word? Because there are two O's, we'll show all the ways to write it out where one O is marked with a bar on top so you can see the differences.

- $\bar{O}OZ$
- $O\bar{O}Z$
- $OZ\bar{O}$
- $\bar{O}ZO$
- $ZO\bar{O}$
- $Z\bar{O}O$

There are 6 different ways to write the word, but if the bar was gone, you'd really think that many of them were already counted. Moving around the two O's in the word doesn't change the word itself. Since there are 2! = 2 ways to rearrange the O's, we should divide the total by that number to find the number of unique ways to write the word ZOO.

Number of actual (unique) ways to rewrite ZOO: $\frac{3!}{2!} = 3$.

Interactive Example: How many unique ways can we rearrange the word SCOTT?

How many total letters are there?___________

Are there any duplicates – and if so, how many different ways can we arrange those letters? _____

The number of unique ways to rewrite SCOTT is ____________

EXPLORE (2)! How many unique ways can we rearrange the following word or words?

A) ** TELEPHONE

B) ** MATHEMATICS

C) **(L)** TREE-HOUSE

D) **(R)** TENNESSEE

E) **(L)** MISSISSIPPI

F) **(R)** BUBBLE-BOBBLE

For these **permutations of like objects**, the order of arrangement matters and we're still selecting all the objects, however some of them are the same. For m objects with n of one type, p of another, and q of a third, the number of unique ways to rearrange is: $\frac{m!}{n! \times p! \times q!}$.

EXPLORE (3)! In more realistic problems, determine the number of ways to rearrange the items.

A) ** How many ways can you arrange 5 books on a shelf if 2 books are identical?

B) How many ways can you arrange 20 DVDs if there are 5 identical copies of Pulp Fiction, 4 identical copies of Avatar, and 3 identical copies of Titanic?

Now let's look at selecting objects where the order doesn't matter. When dealing with sets, the order of elements in the set doesn't matter. So {A, B, C} is the same set as {B, A, C}. Now we'll start with 5 elements and decide to take 3 of them and create a subset. [If the subset idea is too abstract, you could consider 5 shirts that you own and you're going to pack 3 of them for a trip.]

If we arranged these in order (permutation style), there would be ${}_5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = 60$ options. Writing all options out would look like this:

ABC	ABD	ABE	ACD	ACE	ADE	BCD	BCE	BDE	CDE
ACB	ADB	AEB	ADC	AEC	AED	BDC	BEC	BED	CED
CAB	DAB	EAB	DAC	EAC	EAD	DBC	EBC	EBD	ECD
CBA	DBA	EBA	DCA	ECA	EDA	DCB	ECB	EDB	EDC
BAC	BAD	BAE	CAD	CAE	DAE	CBD	CBE	DBE	DCE
BCA	BDA	BEA	CDA	CEA	DEA	CDB	CEB	DEB	DEC

Anytime we select 3 objects, there are 3! = 6 ways to arrange those objects. Since the order doesn't matter to us, all 6 items in a column count as 1 option. Because we're choosing 3 objects, this amounts to taking the number of permutations and dividing by 3!. This new concept is called **combinations** and has the following formula (found again in the PRB menu on TI-30XIIS) or shift ÷ on the Casio fx-300ES Plus:

${}_5C_3 = \frac{{}_5P_3}{3!} = \frac{5!}{3! \times (5-3)!} = \frac{5!}{3! \times 2!} = 10$. In general, ${}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{r! \times (n-r)!}$.

EXPLORE (4)! Compute the following using the combination button on your calculator.

A) ** ${}_7C_4$

B) **(L)** ${}_{52}C_2$ (number of ways to get your first two cards in Blackjack)

C) **(R)** ${}_{51}C_5$ (number of ways to select the final 5 contestants in Miss Teen USA)

EXPLORE (5)! Determine how many different options are possible with the information:

A) ** You have 12 pairs of shoes but can only pack 4. How many options do you have?

B) There are 5 elements in a set. How many subsets can be formed with:

a. 0 elements?

b. 1 element?

c. 2 elements?

d. 3 elements?

e. 4 elements?

f. 5 elements?

C) There are 52 cards in a standard deck. How many 5-card hands are possible?

D) For a lottery, there are 5 numbers chosen from 1 to 69. How many different tickets are possible?

E) For Powerball, there are 5 numbers chosen from 1 to 69, then 1 number chosen from 1 to 26. How many different tickets are possible?

F) For MegaMillions, there are 5 numbers chosen from 1 to 75, then 1 number chosen from 1 to 15. How many different tickets are possible?

G) There are 35 students in a math class, and 3 student names will be drawn from a hat to win a $50 Amazon gift card. How many different ways can the prizes be given out?

H) There are 35 students in a math class and 3 student names will be drawn from a hat to win a prize: $100, $75, or $50. How many different ways can the prizes be given out?

2.15: Summary

Logic concepts:

- Argument fallacies – we saw ways the people can confuse an argument or try to weaken someone's position… and none of them have to do with the quality of the argument that is made.
- Inductive vs. deductive reasoning – we saw that deductive reasoning will have a guaranteed conclusion, but that inductive reasoning can still be useful.
- Logic
    - statements and quantified statements can be looked at to find whether they are true or false.
    - statements can be combined with operators (not, and, or, if-then, iff) to create compound statements, and we saw how to negate each one.
    - ways to rewrite conditional statements to create a different (but equivalent) statement.
    - ways to analyze basic arguments and determine whether it is valid and sound, involving truth tables and Venn diagrams.
- Sets
    - collections of objects have certain properties.
    - Elements and subsets establish relationships, and sets can have operations on them too (not, and, or)
    - We saw how to represent sets visually, and then how to use Venn diagrams to look at the concept of counting how many objects are in each region.

Fortunately, the set operations link very nicely to our logic operators with one small change: set operators are used between sets and logic operators are used between statements.

Logic Name	Slang	Logic Symbol
Negation	Not	$\sim p$
Conjunction	And	$p \wedge q$
Disjunction	Or	$p \vee q$
Conditional	If-then	$p \rightarrow q$

Set Name	Slang	Set Symbol
Complement	Not in B	$\overline{B}$
Intersection	And	$A \cap B$
Union	Or	$A \cup B$
Subset	If-then	$A \subseteq B$

Counting techniques summary (more on next page):

Technique	Can we repeat items?	Are we selecting All or Some?	Does the order of selection matter?	Formula
Fundamental Counting Principle	Yes	All or Some	Yes	$m \times n$
Factorial	No	All only	Yes	$n!$
Permutations	No	All or Some	Yes	${}_nP_r = \frac{n!}{(n-r)!}$
Permutations of like objects	No	All only	Yes	$\frac{m!}{n! \times q! \times p!}$
Combinations	No	All or Some	No	${}_nC_r = \frac{n!}{r! \times (n-r)!}$

- **Fundamental Counting Principle**
  - If you have *m* choices for the first option and *n* choices for the next, then there are $m \times n$ ways to pick one of each. It can be extended to as many options as you want.
- **Factorial**
  - Essentially a special case of permutations where all *n* items are selected, resulting in n! ways.
- **Permutations**
  - The number of ways we can choose and arrange r objects from n objects where the order of arrangement matters is ${}_nP_r$. Remember the following tips:
    - There are *n* different (unique) objects to choose from.
    - We select *r* of the objects (without replacement)
    - We consider arrangements of items to be different: abc is different from bac, bca, etc.
  - The formula for permutations is ${}_nP_r = \frac{n!}{(n-r)!}$.
- **Permutations of like objects**
  - The order of arrangement matters, and we are selecting all of the objects (not just some), but some of the objects can be the same. For *m* total objects, where *n* are of one type, *p* of another type, and *q* of a third type, the number of unique ways to rearrange is: $\frac{m!}{n! \times p! \times q!}$.
- **Combinations**
  - The number of ways we can choose and arrange r objects from n objects where the order of arrangement *does not matter* is ${}_nC_r$. Remember the following tips:
    - There are *n* different (unique) objects to choose from.
    - We select *r* of the objects (without replacement)
    - We consider arrangements of items to be the same: abc is the same arrangement as bac, bca, acb, cab, and cba. When using combinations, all of these are not considered different and we count all of these permutations together as only one combination.
  - The formula for combinations is ${}_nC_r = \frac{n!}{(n-r)! \times r!}$.

INDEX (in alphabetical order):

All 17
argument 7
bi-conditional 28
cardinal number 51
combinations 71
complement 51
compound statements 21
conclusion 7, 23, 33
condition 23
conditional 23
conjunction 21
Contrapositive 23
Converse 23
De Morgan’s Laws 24
deductive reasoning 14
disjoint 54
disjunction 22
elements 48
empty set 49
equal sets 50
factorial 65
Fallacies related to assumptions 11
Fallacies related to relevance 8
fallacy 7, 33
Fundamental Counting Principle 64
inductive reasoning 14
intersection 54
invalid argument 33
Inverse 23
logic 7
logically equivalent 23
logically equivalent statements 20
Mastermind 44
Minesweeper 3
mutually exclusive 54
negation 18
None 17
not sound 41
one-to-one correspondence 65
permutations 67
permutations of like objects 70
premises 7, 33
propositions 17
quantifiers 17
Scientific Method 16
sets 48
Some 17
sound 41
statements 17
subset 49
Truth Table 20
union 55
universe 51
valid argument 7, 33
Venn diagrams 37
Wason Selection Test 30