

Math Fundamentals for Statistics I (Math 52)



Unit 6: Rates, Ratios, and Proportions

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“The ‘How’ and ‘Whys’ Guys”



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6.1: Comparing Objects – Ratios and Rates

When baking crepes, a recipe called for 200 grams of milk and 100 grams of flour. However, if you wanted to make more or less, you could change these amounts and preserve the same relationship between the amounts. If we wanted to put in 100 grams of milk, we would need 50 grams of flour. Comparing these quantities creates our next concept.

With any two objects, the units measuring that object are important and can be used to help us understand the relationship between the objects. If the numbers being compared have the same units of measure, then the comparison is called a **ratio**. Ratios would be statements like 2 to 1, 10 to 1, or 5 to 2, and might also be written in fraction form or with a colon. The ratio 5 to 2 means the same thing as $\frac{5}{2}$ or 5:2. Using a ratio, without units, indicates that the ratio will hold true when the units for numerator and denominator are the same. 5:2 could be 5 cups to 2 cups, or 5 feet to 2 feet, or even 5 people to 2 people.

Example (1): Find the ratio in the crepe recipe.

Solution: Since there are 200 grams of milk and 100 grams of flour, the units are the same (both measure grams). So the ratio would be 200 to 100 or $\frac{200}{100}$. And if we simplified this with FLOF, we'd end up with a ratio of $\frac{200}{100} = \frac{2}{1}$, or 2:1. The ratio would allow us to put in 2 cups of milk for 1 cup of flour, or 2 ounces of milk for 2 ounces of flour. The fate of our crepes is within our hands!

But what happens when the units are different – like number of eggs to cups of flour? When the numbers being compared have different units of measure, the comparison is called a **rate**. Because there are different units, the rate must include the units. When we measure a rate of miles per gallon, we are comparing two quantities; 50 mpg means that a car would be able to go 50 miles for each gallon, and we read this as 50 miles-per-gallon. Anytime we see **per**, we can replace it with “for each,” or vice-versa. To connect the concept with a ratio, we could write a rate in fraction form as $\frac{50 \text{ miles}}{1 \text{ gallon}}$. In order to find equivalent rates, we could use FLOF again.

Example (2): Which of the following have equivalent rates to $\frac{50 \text{ miles}}{1 \text{ gallon}}$?

- | | |
|--------------------------|----------------------------|
| A) 50 miles per gallon | D) 100 miles on 2 gallons |
| B) 50 gallons per mile | E) 5 gallons for 250 miles |
| C) 1 gallon for 50 miles | F) 4 miles for 200 gallons |

Solution: Using FLOF, $\frac{50 \text{ miles}}{1 \text{ gallon}}$ would be equivalent to $\frac{100 \text{ miles}}{2 \text{ gallons}} = \frac{250 \text{ miles}}{5 \text{ gallons}}$; parts A, C, D, and E would be equivalent.

EXPLORE (1)! Find different ways to represent \$30 per hour as a rate.

- A) ** 0.50 dollars per minute
 $\$30 \text{ per } 60 \text{ min} = \0.50 per min
- C) 50 cents per minute
- B) 240 dollars per day (8-hour day)
- D) 7 hours for \$210

For many ratios and rates, it is extremely helpful to see the comparison linked with a denominator of 1. Instead of \$20 for 8 pounds of fish, we would write it as \$2.50 for 1 pound, or \$2.50 per pound. Writing rates in this way for just one comparison is called a **unit rate**.

The way to write a value as a unit rate is to either simplify, with FLOF like fractions, or with division.
 $\frac{\$20}{8 \text{ pounds}}$ would be $20 \div 8 = 2.50$, and then add the units to end with \$2.50.

In the last set of problems, we have to make a determination on what version sounds better: 'per' or 'for each.' Saying 7 hours per \$50 doesn't seem to sound as nice as 7 hours for each \$50. Perhaps even just saying 7 hours for \$50 would be ok.

Interactive Example (3): As a class, come up with different types of ratios and rates that you've seen outside the classroom. Make sure you state whether the comparison is a ratio or a rate!

Answers vary:

miles per gallon

gallons per \$

bananas per dollar

cents per ounce.

EXPLORE (2)! On a sign at McDonalds, the price for the meal (burger, fries, and drink) displayed a savings over the items purchased separately. The sign stated the savings was 0.55¢ per meal.

Explain whether you agree with the sign or not, and why.

Bit of a problem: 55¢ savings makes sense.
But 0.55¢ is a bit more than half of a cent,
which doesn't make sense!

EXPLORE (3)! Write the following as unit rates. It may be helpful to know that 1 pound = 16 ounces. Round any decimals to 4 decimal places.

	Quantity	Unit Rate Desired	Rate (Fraction)
A) **	\$2 for 5 pounds	0.4 dollars per pound	$\frac{2}{5}$ dollars per pound
B)	\$2 for 5 pounds ↳ 200¢ for 5 pounds	40 cents per pound	40 cents per pound
C)	\$2 for 5 pounds ↳ 80 ounces	0.025 dollars per ounce	$\frac{1}{40}$ dollars per ounce
D)	\$2 for 5 pounds ↳ 200¢ for 80 ounces	2.5 cents per ounce	$\frac{5}{2}$ dollars per ounce
E)	\$2 for 5 pounds	2.5 pounds per dollar	$\frac{5}{2}$ pounds per dollar
F) **	\$1.45 for 200 napkins	0.00725 dollars per napkin	$\frac{29}{4000}$ dollars per napkin
G)	\$1.45 for 200 napkins	137.931 ⁰³ napkins per dollar	$\frac{4000}{29}$ napkins per dollar
H)	\$1.45 for 200 napkins	1.37931 napkins per penny	$\frac{40}{29}$ napkins per penny
I)	450 miles on 14.2 gallons of gas	31.69014 miles per gallon	$\frac{2250}{71}$ miles per gallon
J)	450 miles on 14.2 gallons of gas	0.03156 gallons per mile	$\frac{71}{2250}$ gallons per mile

EXPLORE (4)! In the game of Roulette, a wheel with 38 numbered regions is spun to determine the winning number. If you make a bet on ...

A) 12 numbers, what is the ratio of winning numbers to losing numbers?

$$12:26 \text{ or } 6:13 \text{ (simplified)}$$

B) 12 numbers, what is the ratio of winning numbers to the total?

$$12:38 \text{ or } 6:19 \text{ (simplified)}$$

C) 2 numbers, what is the ratio of winning numbers to losing numbers?

$$2:36 \text{ or } 1:18 \text{ (simplified)}$$

D) 2 numbers, what is the ratio of winning numbers to the total?

$$2:38 \text{ or } 1:19 \text{ (simplified)}$$

6.2: Proportions

Ratios and rates are helpful to understand how quantities are related to each other, but often times we'd like to create a new ratio that is equivalent to a ratio we already have. In the case of baking, something like the following example comes to mind.

Interactive Example (1): Remember the crepes from the previous section? The recipe called for 200 grams of milk for 100 grams of flour which would produce the good texture. Too much flour and the crepe batter gets lumpy and thick while too much milk and the batter is soupy and runny. Determine the texture if we did the following:

	Change to Original Recipe (Description)	Milk	Flour	M:F Ratio	Texture Description		
	No Change	200 g	100 g	2 : 1	Runny	Good	Lumpy
A) **	Take out 50 grams of each	150 g	50 g	3:1	Runny	Good	Lumpy
B) **	Put in 100 grams of each	300g	200g	3:2	Runny	Good	Lumpy
C)	Double each amount	400g	200g	2:1	Runny	Good	Lumpy
D)	Take out 75 grams of each	125	25	5:1	Runny	Good	Lumpy
E)	Changed to cups	4 cups	2 cups	2:1	Runny	Good	Lumpy
F)	Took out 1 cup from each	3 cups	1 cup	3:1	Runny	Good	Lumpy

When baking, the batter will have the same consistency if the ratios are the same between ingredients. The ratio of milk to flour was 2:1, so from our table, the texture should be “good” for all the times when the ratio was 2:1.

In mathematics, when two ratios or rates are the same, we call them **proportionate**. And when we can write an equal sign between the ratios, like $\frac{200}{100} = \frac{400}{200}$, we call the resulting equation is called a **proportion**. We can confirm these are equal using FLOF, or by rewriting as unit fractions.

Example (2): A bag of Krusteaz pancake mix indicates that a serving size is 2 pancakes, and that it takes 3 cups of mix to make 18 pancakes. How many cups of mix should you put in if you want to make pancakes for 12 people?

Solution: 3 cups of mix makes 18 pancakes, and this forms a rate of 6 pancakes per cup of mix, as a unit rate. Each person eats 2 pancakes, so this rate could be rewritten as 3 people per cup of mix. Since we want enough for 12 people, this last rate is multiplied by 4 and the final result is 12 people per 4 cups of mix.

Unit rates are very helpful in solving these ratio/rate problems, but when we check the results, we can set up the original ratio and the new ratio to see if they are equivalent. For our values, this would be people per cups of mix:

$$\frac{\text{people}}{\text{cups of mix}} = \frac{3}{1} = \frac{12}{4}$$

Without a calculator, find the result of the right hand ratio: $12 \div 4$. Does it match the 3:1 ratio of the left hand side?

Consider a generic proportion, $\frac{a}{b} = \frac{c}{d}$. We will use FLOF to rewrite each side to have common denominators. $\frac{a}{b} = \frac{c}{d}$ becomes $\frac{ad}{bd} = \frac{bc}{bd}$. Since the denominators are the same on both sides, we can look at the numerators to see if they are equal too: so we check $ad = bc$.

Some teachers noticed this and created a shortcut to determine if the proportion is true. Let's see if you can do it yourself: create a rule about how to check a proportion quickly without using unit rates:

$$\frac{a}{b} = \frac{c}{d} \text{ is a true proportion if and only if... } ad = bc$$

▣ **EXPLORE (1)!** Using your rule from above, determine if the following are true proportions.

A) ** $\frac{11}{13} = \frac{51.7}{61.1} \rightarrow 672.1 = 672.1 \checkmark$
True

C) $\frac{7}{\left(\frac{2}{3}\right)} = \frac{\left(\frac{21}{4}\right)}{\left(\frac{1}{2}\right)} \rightarrow 3.5 = 3.5 \checkmark$
True

B) $\frac{7}{9} = \frac{23}{28} \rightarrow 196 \neq 207$
False

D) $-\frac{8}{7} = -\frac{9}{8} \rightarrow 64 \neq 63$
False

Now with proportions, often we are missing one value, but we know the other three. A recipe calls for 6 cups of flour and 4 eggs, how many eggs should we put in once we realize we only have 3 cups of flour? This type of question can be used to create a proportion, with one missing piece. When there is an unknown quantity, we can still deal with the proportion with a **variable**. A variable is a symbol or letter that represents an unknown number or quantity.

Example (3): Write the previous recipe story as a proportion.

Solution: There are a few ways to set up these ratios, but all of them will use the same variable quantity. We are missing the number of eggs to put with 3 cups of flour, so we start by defining our variable.

Let x represent the number of eggs needed with 2 cups of flour.

Now we'll build a proportion using the rate "cups of flour" to "eggs." $\frac{\text{cups of flour}}{\text{eggs}}$

$$\frac{6}{4} = \frac{3}{x}$$

recipe side what we have

We could have built one using the rate "eggs" to "cups of flour." $\frac{\text{eggs}}{\text{cups of flour}}$ $\frac{4}{6} = \frac{x}{3}$

recipe side what we have

We could have built a proportion using the rate "recipe" to "we have." $\frac{\text{recipe}}{\text{we have}}$ $\frac{4}{x} = \frac{6}{3}$

egg side flour side

Interactive Example (4): Using the 3 proportions from above, multiply them out to rewrite without fraction form. The first is done for you, but try the other proportions in the space below.

The first proportion $\frac{6}{4} = \frac{3}{x}$ would be written as $6x = 12$.

$$\frac{4}{6} = \frac{x}{3} \quad \text{is written as } 12 = 6x$$

$$\frac{4}{x} = \frac{6}{3} \quad \text{is written as } 12 = 6x$$

all are the same!

No matter how we write the proportion, if we keep the units correctly matched, the same equation would result when we check the proportion. But to solve proportions, to find the missing value, we'll need a different process. To solve proportions, we'll use a handy technique involving unit rates.

Solving Proportions Algorithm: Our example will be $\frac{5}{12} = \frac{3}{x}$.

Step 1: Write your proportion out.

Step 2: The technique we'll use is to rewrite proportions so the variable is on the top. We can take the reciprocal of both sides, so $\frac{5}{12} = \frac{3}{x}$ becomes $\frac{12}{5} = \frac{x}{3}$. If the proportion started with the variable on top, so much the better – we can skip this step!

Step 3: Create a unit rate out of the side that does not have a variable, using your calculator. In this case, we'd type in $12 \div 5 = 2.4$. While we don't have to write this step down, it can be helpful. We now have $\frac{12}{5} = \frac{x}{3}$ becoming $\frac{2.4}{1} = \frac{x}{3}$.

Step 4: Use FLOF to scale this up. Since the right hand side is x over 3, we'll FLOF by 3. Our proportion $\frac{2.4}{1} = \frac{x}{3}$ becomes $\frac{2.4(3)}{1(3)} = \frac{x}{3}$, which simplifies to $\frac{7.2}{3} = \frac{x}{3}$.

We can now read off the value of x through the other numerator, which is 7.2.

<i>Step 1</i>	<i>Step 2</i>	<i>Step 3</i>	<i>Step 4</i>
$\frac{5}{12} = \frac{3}{x}$	$\frac{12}{5} = \frac{x}{3}$	$\frac{2.4}{1} = \frac{x}{3}$	$\frac{2.4(3)}{1(3)} = \frac{x}{3}$ becomes $\frac{7.2}{3} = \frac{x}{3}$

Example (5): ■ Solve the proportion $\frac{50}{19} = \frac{40}{x}$.

- Start with the initial equation: $\frac{50}{19} = \frac{40}{x}$
- Then take reciprocals: $\frac{19}{50} = \frac{x}{40}$
- Now scale up the ratio: $\frac{19}{50} = \frac{x}{40} \Rightarrow 40\left(\frac{19}{50}\right) = x$
- Simplify by hand or using calculator: $x = 40\left(\frac{19}{50}\right) = 15.2$
- Once done, we should check the result: $\frac{50}{19} = \frac{40}{15.2} \Rightarrow 50(15.2) = 760$ and $40(19) = 760$

It all checks out, so we know that $x = 15.2$ is the solution.

EXPLORE (2)! Using the rule from the previous page to solve the following proportions. Keep the end results the same as the starting – if you start with fractions/decimals, end with them.

A) ** $\frac{x}{7} = \frac{2.1}{\left(\frac{24}{5}\right)} \rightarrow \frac{x}{7} = \frac{0.75}{1}$
 $\rightarrow \frac{x}{7} = \frac{0.75(7)}{1(7)} = \frac{5.25}{7}$
 $x = 5.25$

C) $\frac{7}{15} = \frac{28}{x} \rightarrow \frac{x}{28} = \frac{15}{7} \rightarrow \frac{x}{28} = \frac{15}{7}$
 $\rightarrow \frac{x}{28} = \frac{15(28)}{7(28)} = \frac{60}{28}$
 $x = 60$

B) $\frac{9}{11} = \frac{13}{x} \rightarrow \frac{x}{13} = \frac{11}{9} = \left(\frac{11}{9}\right)$
 $\rightarrow \frac{x}{13} = \frac{11(13)}{9(13)} = \frac{143}{9}$
 $x = \frac{143}{9}$

D) $\frac{x}{\left(\frac{9}{11}\right)} = \frac{\left(\frac{22}{7}\right)}{\left(\frac{3}{2}\right)} \rightarrow \frac{x}{\left(\frac{9}{11}\right)} = \frac{44}{21} \left(\frac{9}{11}\right)$
 $\rightarrow \frac{x}{\left(\frac{9}{11}\right)} = \frac{12}{7}$
 $x = \frac{12}{7}$

Example (6): Proportions come up often in real life. Here's one:

A doctor prescribes 14 mg of medicine for every 25 pounds of body weight. How much medicine would we give to a child weighing 89 pounds? (round to 1 decimal place)

Set up a proportion of mg medicine to pounds of body weight. $\frac{14}{25} = \frac{x}{89}$. Then we solve for the

missing piece: $\frac{14}{25} = \frac{x}{89} \Rightarrow x = 89 \left(\frac{14}{25}\right) = 49.84$. So we need to give the child 49.84 mg of medicine!

EXPLORE (3)! Here's a few problems from real life where proportions could come in handy. For each, set up a proportion, then use our technique to solve it!

A) A Poulan chainsaw requires a gas-oil mix, in a ratio of 40:1. How many fluid ounces of oil do you need to add to one gallon of gasoline? [Note: One gallon contains 128 fluid ounces.]

$\frac{\text{gas}}{\text{oil}} \rightarrow \frac{40}{1} = \frac{128}{x} \rightarrow \frac{x}{128} = \frac{1}{40} = \left(\frac{1}{40}\right)$
 $\frac{x}{128} = \frac{1(128)}{40(128)} = \frac{3.2}{128}$
 $x = 3.2 \text{ fluid ounces}$

B) Yesenia looks to buy a home valued at \$640,000 but wants to know the annual property taxes. The neighbor pays \$7825 per year on a home value of \$700,000. How much would Yesenia pay in property taxes each year?

$\frac{7825}{700,000} = \frac{x}{640,000} \rightarrow \frac{x}{640,000} = \frac{7825(640,000)}{700,000}$
 $x = 7,154.29$

6.3: Percents

Perhaps one of the most common ratios used outside of math classrooms is one that affects money, interest rates, student loans, mortgages, and more! Remember that a ratio is a comparison of two objects.

Example (1): Rewrite the ratio 6 for every 10 in many ways.

Thinking of this with FLOF, we can multiply each by the same number and have an equivalent ratio.

- 6 for every 10
- 3 for every 5
- 12 for every 20
- 30 for every 50
- 60 for every 100
- 120 for every 200
- 300 for every 500
- 600 for every 1000

Since there are so many ways to write a ratio, one was selected as the method of choice. Similar to how scientific notation was chosen as the method to write very large and very small numbers, one was selected as the standard. The one chosen was 60 for every hundred, or 60 per hundred. This was shortened to 60 **percent**, and often written as 60%.

NOTE: *We use percent now, but this word is very recent. Prior to percent, the words per centum were used, from Latin meaning by the hundred.*

Interactive Example (2): Write 3 for every 6 as a percent.

1 for every 2
50 for every 100 50%

We can convert ratios to percents because they can be written as fractions. This means we can convert fractions, decimals, or ratios into percents, or vice versa. One convenient thought is that 100% means 100 per 100, or the fraction $\frac{100}{100}$, which is 1. Because of the multiplicative identity property, multiplying by 1 won't change the value of the number. Per 100 as a fraction would be hundredths, so this converts nicely to decimal numbers: 0.45 is 45 hundredths, or 45%. We can rewrite % as hundredths any time we want because it just looks different but keeps the same value.

Example (3): Convert $\frac{5}{8}$ into a percent.

Method 1: We don't want to change the value, so we will just multiply by 1. You can use your calculator for the quick division: $\frac{5}{8} = \frac{5}{8} \times 1 = \frac{5}{8} \times 100\% = \frac{5}{8} \times \frac{100}{1}\% = \frac{500}{8}\% = 62.5\% = 62\frac{1}{2}\%$.

Method 2: Rewrite the fraction as a decimal using division, then convert to a percent.

$$\frac{5}{8} = 0.625 = \frac{625}{1,000} = \frac{62.5}{100} = 62.5\%$$

▣ **Interactive Example (4):** Complete the table to show the different ways to write the expression.

	Ratio	Fraction or Mixed Number	Decimal	Percent
A) **	9:20	$\frac{9}{20}$	0.45	45%
B) **	235:100 or 47:20	$\frac{235}{100} = \frac{47}{20}$	2.35	235%
C) **	7:9	$\frac{7}{9}$	$0.\overline{7}$	$77.\overline{7}\%$
D)	73:100	$\frac{73}{100}$	0.73	73%

Create a rule for how to convert a decimal number into a percent:

Move the decimal place two spots to the right and write "%" after.

Create a rule for how to convert a percent into a decimal number:

Move the decimal place two spots to the left and remove the "%" after.

Create a rule for which option you'll use to convert a fraction into a percent:

Rewrite as division, and then divide numerator by denominator.
or multiply by 100%.

▣ **EXPLORE!** Complete the table to show the different ways to write the expression.

	Ratio	Fraction	Decimal	Percent
E)	27:12	$\frac{27}{12} = \frac{9}{4}$	2.25	225%
F)	23:1000	$\frac{23}{1000}$	0.023	2.3%
G)	17:3	$5\frac{2}{3} = \frac{17}{3}$	$5.\overline{6}$	$566.\overline{6}\%$
H)	1:400	$\frac{1}{400}$	0.0025	0.25%

6.4: Solving Percent Problems

Since we've now seen that percentages are a form of ratio, then we can set up proportions to solve for any missing pieces.

Example (1): 35 is 80% of what?

In this case, we don't know what this is out of, so we'll use the variable x for that. Then label all the other pieces quickly. We can set up the proportion to be $\frac{80}{100} = \frac{35}{x}$, or if we prefer: $\frac{100}{80} = \frac{x}{35}$.

Solving the proportion shows $x = 43.75$.

▣ **EXPLORE (1)!** Find the missing value in these percent problems.

A) ** What is 30% of 45

$$\frac{30}{100} = \frac{x}{45}$$

$$x = 13.5$$

D) ** What percent is 90 of 75?

$$\frac{x}{100} = \frac{90}{75}$$

$$x = 120\%$$

B) 85% of 60 is what?

$$\frac{x}{60} = \frac{85}{100}$$

$$x = 51$$

E) 20 is 60% of what?

$$\frac{20}{x} = \frac{60}{100} \rightarrow \frac{x}{20} = \frac{100}{60}$$

$$x = \frac{100}{3} = 33.\bar{3}$$

C) What percent of 80 is 73?

$$\frac{x}{100} = \frac{73}{80}$$

$$x = 91.25$$

F) 125% of what is 54?

$$\frac{54}{x} = \frac{125}{100} \rightarrow \frac{x}{54} = \frac{100}{125}$$

$$x = 43.2$$

Example (2): If you paid \$6,524 in property tax on a home valued at \$584,000, what percent tax did you pay?

Set up the proportion: we paid \$6,524 out of \$584,000 and are missing the percent portion.

$$\frac{x}{100} = \frac{6,524}{584,000} \Rightarrow x = 100 \left(\frac{6,524}{584,000} \right) \approx 1.117123\% \quad \text{The percent tax paid is about 1.12\%}$$

Interactive Example (3): If you paid \$52.08 for an item (with tax included), and the price on the shelf was \$45, what percent of the \$45 did you pay? What was the sales tax rate as a percent?

$$\frac{x}{100} = \frac{52.08}{45}$$

$$\underline{x = 115.7\bar{3}\%}$$

We paid 115.7 $\bar{3}$ % of the \$45,
which is 15.7 $\bar{3}$ % sales tax.

Interactive Example (4): If you bring home \$2,400 per month, and rent is \$960, what percentage of your pay is going to rent?

$$\frac{x}{100} = \frac{960}{2400} \rightarrow \boxed{x = 40\%}$$

In life, we use percentages often, and sometimes they are used to find how much a value changed by increasing or decreasing. We call this the **percent increase** or **percent decrease**. With any value, we always start with 100% of it. Should we add to it, then we'll have more than 100% of the starting value. If we take some away, there will be less than 100% of the starting values. The amount of increase or decrease is fairly easy to find with subtraction; the key to percent problems is that we always use the starting amount as our base and increase or decrease from there.

There are two methods to determine the percent increase/decrease.

Method 1: Value of the Difference (Traditional).

For this example, we'll go from \$40 to \$50 and try to find the percent increase.

Step 1: Compute the difference of End – Start = 50 – 40 = 10.

Step 2: Compute this difference as a percentage of the starting amount. $\frac{x}{100} = \frac{10}{40} \Rightarrow x = 25\%$.

The result shows that the difference is positive 25%, so \$50 is a 25% increase over \$40. This method works all the time, and if the difference is negative, this will be a percent decrease.

Method 2: Compare to 100%.

For this example, we'll use the premise from method 1 and try to speed it up.

$$\frac{\text{Difference}}{\text{Start}} = \frac{\text{End} - \text{Start}}{\text{Start}} = \frac{\text{End}}{\text{Start}} - \frac{\text{Start}}{\text{Start}} = \frac{\text{End}}{\text{Start}} - 1$$

Step 1: Compute the quotient of $\text{End} \div \text{Start} = 50 \div 40 = 1.25$. This says 50 is 125% of 40.

Step 2: Subtract 1 from this result and turn into a decimal: $1.25 - 1 = 0.25 = 25\%$.

The result also shows that the difference is positive 25%, so \$50 is a 25% increase over \$40. We strongly recommend the Method 2 approach. One reason is that you only need to type in the original numbers once, instead of twice like Method 1. Another is that this idea ties into upcoming sections.

Example (5): What is the percent change from 11,392 to 9,844? Round to 2 decimal places.

$$\frac{9,844}{11,392} - 1 \approx 0.864115... - 1 \approx -0.1359. \text{ So the percent change was a decrease of 13.59\%.}$$

EXPLORE (2)! Determine the percent increase or decrease using either method. Round result to 2 decimal places.

	Start	End	Increase or Decrease	Percent
A) **	125	140	Increase	12%
B)	60	20	Decrease	66.7%
C)	20	25	Increase	25%
D)	25	20	Decrease	20%

Based on the exploration above in the last two parts, is it true that the percent increase from start to end is the same as the percent decrease from end to start? Explain why or why not.

We can use this concept to be able to increase/decrease a number if we know the percent increase or percent decrease. If there was a 5% increase on \$60, then we will end up with 105% of the starting amount; as a result, we will multiply the starting number by 1.05 to end with $(1.05)(60) = 63$.

Example (6): If a bottle was priced at \$87, and had a 20% off coupon, what would be the final price?

A 20% off coupon means 20% decrease. Since we are decreasing by 20%, then we will end up with 80% of the starting amount instead of 100%. So we multiply the starting number (87) by 0.80 to end with $(0.80)(87) = 69.6$. The final price would be \$69.60.

■ **EXPLORE (3)!** Use the percent increase or decrease to find the final amount.

- A) ** Property taxes were \$5,020 this year but increased by 7.2%. How much would we pay next year?

$$5020 \overset{107.2\%}{\underbrace{(\quad)}} (1.072) = \boxed{\$5,381.44}$$

- B) ** An item at Disneyland was priced at \$22.20. Once you include the 8.1% sales tax, how much would you pay total?

$$(22.20) \overset{108.1\% \leftarrow \text{increase}}{\underbrace{(\quad)}} (1.081) = \underline{\$24.00} \text{ (rounded to nearest penny)}$$

- C) If Josue was making \$14.12 per hour and earned a 20% increase, how much would he earn per hour after the raise?

$$14.12 \overset{120\%}{\underbrace{(\quad)}} (1.20) = \underline{\$16.94}$$

- D) Faculty members at Palomar were paid \$65,000 per year to start. However, during negotiations, the starting salary was cut by 1.45%; what is the new starting salary?

$$65000 \overset{98.55\%}{\underbrace{(\quad)}} (0.9855) = \underline{\$64,057.50}$$

- E) While at a Kohl's sale, a customer brought in two coupons – one for 20% off, and one for 30% off – to be used one after another on an item that costs \$54.95.

- a. Determine the final price if you used the 20% off coupon first, then the 30% off coupon.

$$(54.95) (0.80) (0.70) = \$30.77$$

- b. Determine the final price if you used the 30% off coupon first, then the 20% off coupon.

$$(54.95) (0.70) (0.80) = \$30.77$$

- c. What property that we have learned guarantees what we just discovered?

The commutative property of multiplication.

6.5: Slope

We graphed a number of lines in Unit 1, and when graphing them, we asked which line was increasing faster. At the time, we only had the ability to gauge this increase or decrease visually, by looking at the steepness of the line.

Mathematicians prefer to have a way to measure concepts like this with numbers, instead of just a feeling. The idea given here is that the steepness of the line describes a rate of change, and this concept is called **slope**. The slope between two points on a line is the ratio of the change in y -coordinates to the change in the x -coordinates. Slope is often represented with the letter m .

$$\text{Slope} = m = \frac{\text{Change in } y \text{ - coordinates}}{\text{Change in } x \text{ - coordinates}}$$

Example: Find the slope of the line passing through (1, 6) and (5, 15).

Look at the y -coordinates first; the y -coordinates moved from 6 to 15. How much did this change? This was an increase of 9, which will be the numerator (+ 9). Next, look at the change in x -coordinates, which moved from 1 to 5. This is an increase of 4, which will be the denominator (+ 4).

So $m = \frac{+9}{+4} = \frac{9}{4}$, and this indicates that the line is increasing by 9 units vertically for every 4 unit

increase horizontally. We could write this as a unit ratio, $9 \div 4 = 2.25$, so the line is increasing by 2.25 units vertically for every 1 unit increase horizontally.

EXPLORE (1)! If possible, find the slope of the line from the two points given; simplify fractions.

A) ** (8,11) and (14,19) $m = \frac{+8}{+6} = \frac{4}{3}$

D) ** (5,-2) and (-1,4) $m = \frac{+6}{-6} = -1$

B) (2,5) and (9,5) $m = \frac{+0}{+7} = 0$

E) (-2,-5) and (-8,-3) $m = \frac{+2}{-6} = -\frac{1}{3}$

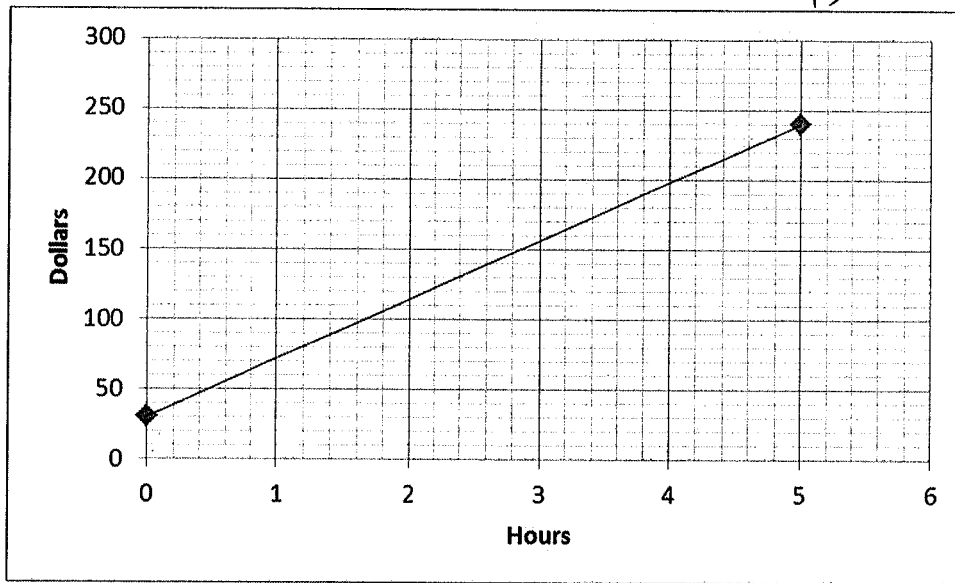
C) (-1,6) and (-1,10) $m = \frac{+4}{+0}$
undefined

F) (-11,-4) and (-14,-10) $m = \frac{-6}{-3} = 2$

Knowing how to find the slope from any two points in this way will help us to determine the slope from just the graph of the line. Remember that the slope is a rate of change, and the units on that rate will be the units of y per unit of x . If the y -coordinate measures miles and the x -coordinate measures gallons, then the slope has units of miles per gallon.

Interactive Example: Here's a graph showing the amount of money charged by a plumber based on the amount of time working on a job.

A) What two points are on the graph? $(0, 30)$ and $(5, 240)$



B) Find the slope of the line from the points on the graph.

$$m = \frac{210}{5} = \frac{42}{1} \text{ or } 42$$

C) ** Interpret this slope using a sentence and the appropriate units.

The plumber charges \$42 per hour as his rate.

D) ** Interpret the two points using sentences.

$(0, 30)$ means the plumber charges \$30 for 0 hours.

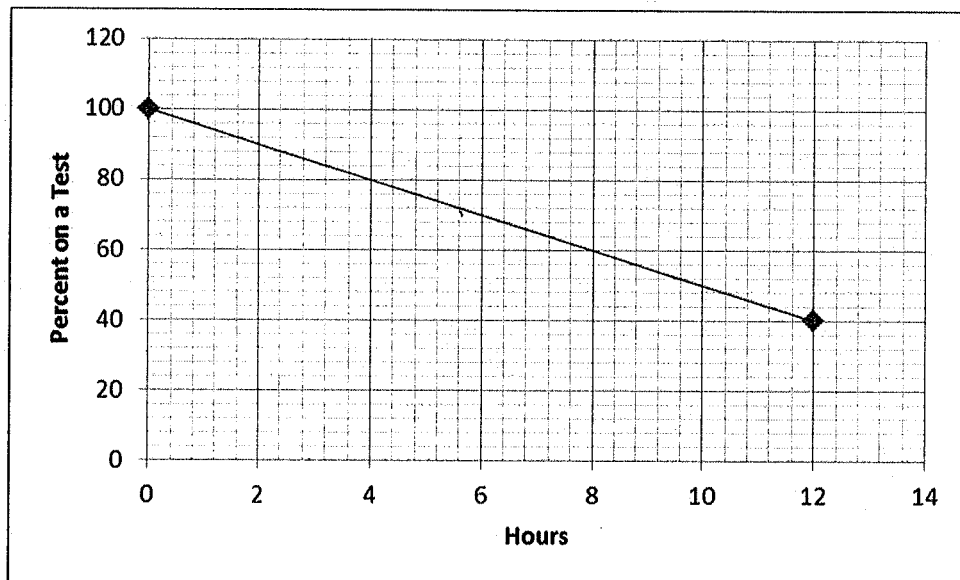
$(5, 240)$ means the plumber charges \$240 for a 5 hour.

E) Using the graph, about how much would the plumber charge for 2 hours of work?

From the graph, it will be a charge of about \$115.

■ **EXPLORE (2)!** Determine the slope of the line from the graph, and interpret the slope with a sentence that includes the appropriate units.

A) This graph shows the grade on a test (percent) based on how long a student was on their cell phone the night before the test.



$$\begin{array}{l} (0, 100) \\ +12 \downarrow \quad \downarrow -60 \\ (12, 40) \end{array}$$

$$m = \frac{-60}{12} = \underline{-5}$$

Slope: -5 Interpretation: The percent on the test goes down by 5% for each hour on their cell phone.

Someone on their phone for 5 hours and 45 minutes would earn what grade on the test (use the graph)?

About 72%.

Someone who turned their phone off and studied for the exam would earn what grade on the test (use the graph)?

With 0 hours of cell phone use, the grade would be 100. ☺

For Love of the Math: There are more complicated formulas for this, like $m = \frac{y_2 - y_1}{x_2 - x_1}$, but the

concept remains the same. It still represents the rate of change measured by "change in y" over "change in x." For many students, working to understand the concept might be better than working to just memorize formulas, and many students make mistakes in this formula with subtraction and negative numbers.

6.6: Dimensional Analysis and Unit Conversions

When we are given one type of unit, like feet or inches, and want to change to another, we use a process known as **dimensional analysis**. This process looks like multiplying fractions and requires some knowledge of the sizes of different units of measurement. Some common units and sizes are stated below for both the US system and the Metric system:

Distance	Area	Volume	Weight/Mass
1 foot = 12 inches	1 ft ² = 144 in ²	1 gallon = 231 in ³	1 lb = 16 oz
1 yard = 3 feet	1 yd ² = 9 ft ²	1 gallon = 128 fl. oz.	1 ton = 2,000 lb
1 mile = 5,280 feet	1 acre = 43,560 ft ²	1 L = 1000 mL	1 kg ≈ 2.204 lb
1 inch = 2.54 cm	1 cm ² = 100 mm ²	1 gal = 4 qt	1 g = 1,000 mg
1 km = 1,000 m	1 m ² = 10,000 cm ²	1 qt = 2 pt	1 kg = 1,000 g

In the metric system, water has some excellent properties for conversion purposes: 1 cm³ = 1 mL = 1 g. Sometimes, as in medicine, 1 cm³ is written as 1 cc (cubic centimeters).

Interactive Example (1): Create some conversions of our own using time:

A) 1 minute = 60 seconds

C) 1 year ≈ 52 weeks

B) 1 hour = 60 minutes

D) 1 week = 7 days

For Love of the Math: *When we change from measuring one type of unit to another, we are really converting these units and the process is called **unit conversions**. However, in mathematics, this has a bit of a double meaning. Often in mathematics, we use the word **unit** to represent a value or measurement of 1. Some examples include a unit circle with a radius of 1, a unit vector with a magnitude of 1, and a unit rate with a denominator of 1. For all unit conversions, we will be multiplying by 1, in one form or another!*

Review: What is the multiplicative identity **element**?

1

Review: What is the multiplicative identity **property**?

$$a \times 1 = a$$

A) From the table above, what is the value of $\frac{1 \text{ yard}}{3 \text{ feet}}$? What about $\frac{3 \text{ feet}}{1 \text{ yard}}$?

$\underbrace{\hspace{10em}}_1$

B) Explain what happens when we create a ratio with any two items that are equal.

The ratio is 1.

Example (2): Convert from one unit to another: How many inches in a mile? (*From a S. Gomez Song*)

We start with our initial value, 1 mile, and write it as a fraction: $\left(\frac{1 \text{ mile}}{1}\right)$. Then, we use some of the conversions to multiply by 1 by writing a conversion equation in fraction form. The key idea to understanding where to put each value is to consider what quantity you want to move away from: if we want to move away from miles, we would put miles on the bottom. While we don't have any equations from miles to inches, we could move from miles to feet. Write this value next to the original: $\left(\frac{1 \text{ mile}}{1}\right)\left(\frac{5,280 \text{ ft}}{1 \text{ mile}}\right)$.

If we stopped here, the mile unit would cancel and leave us with feet. However, we could continue on to turn feet into inches, and we'd write that value next to the others: $\left(\frac{1 \text{ mile}}{1}\right)\left(\frac{5,280 \text{ ft}}{1 \text{ mile}}\right)\left(\frac{12 \text{ in}}{1 \text{ ft}}\right)$.

Now we'll multiply the values out and cancel out any units that are in both numerator and denominator: $\left(\frac{1 \cancel{\text{mile}}}{1}\right)\left(\frac{5,280 \cancel{\text{ft}}}{1 \cancel{\text{mile}}}\right)\left(\frac{12 \text{ in}}{1 \cancel{\text{ft}}}\right) = 63,360 \text{ in}$. So this shows 1 mile = 63,360 in; once again, we keep the same value but change the way it looks. Tell me something I do know, Selena!

EXPLORE (1)!

A) ** How many feet are in 6 miles?

$$\left(\frac{6 \text{ mi}}{1}\right)\left(\frac{5280 \text{ ft}}{1 \text{ mi}}\right) = \boxed{31,680 \text{ ft}}$$

B) How many inches are in 6 miles?

$$\left(\frac{6 \text{ mi}}{1}\right)\left(\frac{5280 \text{ ft}}{1 \text{ mi}}\right)\left(\frac{12 \text{ in}}{1 \text{ ft}}\right) = \boxed{380,160 \text{ inches}}$$

C) How many inches are in 1952 yards?

$$\left(\frac{1,952 \text{ yd}}{1}\right)\left(\frac{3 \text{ ft}}{1 \text{ yd}}\right)\left(\frac{12 \text{ in}}{1 \text{ ft}}\right) = \boxed{70,272 \text{ inches}}$$

D) ** How many feet are in 600 cm?

$$\left(\frac{600 \text{ cm}}{1}\right)\left(\frac{1 \text{ yd}}{2.54 \text{ cm}}\right)\left(\frac{1 \text{ ft}}{12 \text{ in}}\right) \approx \boxed{19.69 \text{ ft}}$$

E) How many square inches are in 60 square feet?

$$\left(\frac{60 \text{ ft}^2}{1}\right) \left(\frac{12 \text{ in}}{1 \text{ ft}}\right) \left(\frac{12 \text{ in}}{1 \text{ ft}}\right) = \boxed{8,640 \text{ in}^2}$$

F) How many square yards are in 60 square feet?

$$\left(\frac{60 \text{ ft}^2}{1}\right) \left(\frac{1 \text{ yd}}{3 \text{ ft}}\right) \left(\frac{1 \text{ yd}}{3 \text{ ft}}\right) \approx \boxed{6.67 \text{ yd}^2}$$

G) How many square yards are in 50,000 square ft?

$$\left(\frac{50,000 \text{ ft}^2}{1}\right) \left(\frac{1 \text{ yd}}{3 \text{ ft}}\right) \left(\frac{1 \text{ yd}}{3 \text{ ft}}\right) \approx \boxed{5,555.56 \text{ yd}^2}$$

H) ** How many cubic feet are in 600 square yards?

3-dimensional ← 2-dimensional.

This cannot be done. To change units,
we can only convert to a similar type.

Remember that when converting units, we never change the value – but we do make it look different.

Concept questions:

A) Converting from a larger unit to smaller unit, then the number will look larger ?

B) Converting from a smaller unit to larger unit, then the number will look smaller ?

▣ **EXPLORE (2)!** Convert units using the technique shown here.

A) How many seconds are in 30 days?

$$\left(\frac{30 \text{ days}}{1}\right) \left(\frac{24 \text{ hrs}}{1 \text{ day}}\right) \left(\frac{60 \text{ min}}{1 \text{ hr}}\right) \left(\frac{60 \text{ sec}}{1 \text{ min}}\right) = \boxed{2,592,000 \text{ seconds}}$$

B) ** How many seconds are in one year?

$$\left(\frac{1 \text{ year}}{1}\right) \left(\frac{365 \text{ days}}{1 \text{ year}}\right) \left(\frac{24 \text{ hrs}}{1 \text{ day}}\right) \left(\frac{3600 \text{ sec}}{1 \text{ hr}}\right) = \boxed{31,536,000 \text{ seconds}}$$

C) If you waited for 100,000 seconds, how many days is this?

$$\left(\frac{100,000 \text{ sec}}{1}\right) \left(\frac{1 \text{ hr}}{3600 \text{ sec}}\right) \left(\frac{1 \text{ day}}{24 \text{ hr}}\right) \approx \boxed{1.16 \text{ days}}$$

D) One plot of land is a rectangle measuring 7,200 yards by 3,500 yards. How many acres is this?

$$7200 \times 3500 = 25,200,000 \text{ yd}^2$$

$$\left(\frac{25,200,000 \text{ yd}^2}{1}\right) \left(\frac{3 \text{ ft}}{1 \text{ yd}}\right) \left(\frac{3 \text{ ft}}{1 \text{ yd}}\right) \left(\frac{1 \text{ acre}}{43,560 \text{ ft}^2}\right) \approx \boxed{5,206.61 \text{ acres}}$$

E) ** How many km in 1 mile?

$$\left(\frac{1 \text{ mi}}{1}\right) \left(\frac{5280 \text{ ft}}{1 \text{ mi}}\right) \left(\frac{12 \text{ in}}{1 \text{ ft}}\right) \left(\frac{2.54 \text{ cm}}{1 \text{ in}}\right) \left(\frac{1 \text{ m}}{100 \text{ cm}}\right) \left(\frac{1 \text{ km}}{1000 \text{ m}}\right) \approx \underline{1.61 \text{ km}}$$

F) How many miles in 1 km?

$$\left(\frac{1 \text{ km}}{1}\right) \left(\frac{1000 \text{ m}}{1 \text{ km}}\right) \left(\frac{100 \text{ cm}}{1 \text{ m}}\right) \left(\frac{1 \text{ in}}{2.54 \text{ cm}}\right) \left(\frac{1 \text{ ft}}{12 \text{ in}}\right) \left(\frac{1 \text{ mi}}{5280 \text{ ft}}\right) \approx \underline{0.62 \text{ mi}}$$

We can approximate with $1 \text{ mi} = 1.61 \text{ km}$

$$\left(\frac{1 \text{ km}}{1}\right) \left(\frac{1 \text{ mi}}{1.61 \text{ km}}\right) \approx 0.62 \text{ mi}$$

G) If one US bill has a mass of 1 gram, how much would \$16 million weigh if it was \$20 bills?

$$\left(\frac{\$16,000,000}{1}\right) \left(\frac{1 \text{ gram}}{\$20}\right) \left(\frac{1 \text{ kg}}{1000 \text{ g}}\right) \left(\frac{2.204 \text{ lb}}{1 \text{ kg}}\right) \approx \underline{1763.20 \text{ pounds}}$$

H) ** If one US bill has a mass of 1 gram, how much would \$16 million weigh if it was \$100 bills?

$$\left(\frac{\$16,000,000}{1}\right) \left(\frac{1 \text{ gram}}{\$100}\right) \left(\frac{1 \text{ kg}}{1000 \text{ g}}\right) \left(\frac{2.204 \text{ lb}}{1 \text{ kg}}\right) \approx \underline{352.64 \text{ pounds}}$$

I) In the movie "Assassins" with Sylvester Stallone, he took \$16 million in one briefcase and lifted it easily with one hand. Do you feel this could really happen?

No - one handed 353 pounds... no way.

J) In the movie "The Taking of Pelham 1-2-3" with Denzel Washington, John Travolta demanded \$10 million. When told it was too heavy, he stated that it would be 220 pounds. Determine if this is accurate.

$$\left(\frac{\$10,000,000}{1}\right) \left(\frac{1 \text{ gram}}{\$100}\right) \left(\frac{1 \text{ kg}}{1000 \text{ g}}\right) \left(\frac{2.204 \text{ lb}}{1 \text{ kg}}\right) \approx 220.4 \text{ pounds.}$$

Wow! An accurate movie!

K) If we were assigned a drug at a rate of 50 grams per day, how much is this as mg/hour?

$$\left(\frac{50 \text{ g}}{1 \text{ day}}\right) \left(\frac{1000 \text{ mg}}{1 \text{ g}}\right) \left(\frac{1 \text{ day}}{24 \text{ hrs}}\right) \approx \boxed{2083.33 \text{ mg/hr}}$$

L) ** The fastest human, Usain Bolt, ran 100 meters in 9.58 seconds. What would this be in miles per hour?

$$\left(\frac{100 \text{ m}}{9.58 \text{ s}}\right) \left(\frac{100 \text{ cm}}{1 \text{ m}}\right) \left(\frac{1 \text{ in}}{2.54 \text{ cm}}\right) \left(\frac{1 \text{ ft}}{12 \text{ in}}\right) \left(\frac{1 \text{ mi}}{5280 \text{ ft}}\right) \left(\frac{3600 \text{ sec}}{1 \text{ hr}}\right) \approx \boxed{23.35 \text{ miles per hour}}$$

6.7: Applications

Unit Rate Applications:

- A) ** Costco sells Kirkland Signature bacon containing two 1.5 pound packages, for \$10.95. They also sell Farmer's Maple bacon one 4 pound package for \$15.39. Which one is the better per unit price? Which one would you buy?

$$\frac{\$10.95}{3 \text{ lb}} = \$3.65 \text{ per pound} \leftarrow \text{better unit price.}$$

$$\frac{\$15.39}{4 \text{ lb}} = \$3.84 \text{ per pound}$$

Answers vary on which to choose.

- B) Costco sells a 10 pound bag of granulated sugar for \$4.79, a 25 pound bag for \$11.59, and a 50 pound bag for \$22.49. Which of the three options is the best unit price? Which would you buy if you needed to stock up on some sugar?

$$\frac{\$4.79}{10 \text{ lb}} = \$0.479 \text{ per pound}$$

$$\frac{\$11.59}{25 \text{ lb}} = \$0.4636 \text{ per pound}$$

$$\frac{\$22.49}{50} = \$0.4498 \text{ per pound}$$

best unit price.

I would buy this, as 25 lbs or more is not going to be used quickly.

- C) Costco sells Chobani yogurt containing fifteen 6 ounce cups for \$13.49 (5 strawberry, 5 peach, 5 blueberry). Vons is selling individual 6 ounce cups for 95¢ each. Which one is the better per unit price? Which one would you buy?

$$\frac{\$13.49}{15 \text{ cups}} = \$0.8993 \text{ per cup} \approx \$0.90 \text{ per cup} \leftarrow \text{better unit price.}$$

→ 95¢ per cup

I would buy this because I can pick my flavors ... my kids don't like blueberry.

Vehicle Applications:

- A) ** A new Toyota Avalon claims 31 mpg for highway driving. How many gallons are used in 100 miles? [This is called "gallons per hundred miles"]

$$\frac{31 \text{ miles}}{1 \text{ gallon}} = \frac{100 \text{ miles}}{x} \rightarrow \frac{x}{100} = \frac{1(100)}{31(100)} = \frac{\frac{1}{31}(100)}{(100)} \Rightarrow 3.2258 \approx x$$

3.23 gallons per hundred miles

- B) A new Toyota Tacoma claims 25 mpg for highway driving. How many gallons are used in 100 miles?

$$\frac{25 \text{ mi}}{\text{gal}} = \frac{100 \text{ mi}}{x} \rightarrow \frac{x}{100} = \frac{\frac{1}{25}(100)}{1(100)} = 4 \Rightarrow x = 4$$

4 gallons per hundred miles

- C) Over the course of 12,000 miles (about one year of driving), which vehicle has lower gasoline costs, and how much less? (assume gas is \$3.15 per gallon)

$$\text{Avalon: } \left(\frac{3.23 \text{ gal}}{100 \text{ mi}} \right) \left(\frac{12,000 \text{ mi}}{1} \right) \left(\frac{\$3.15}{\text{gal}} \right) \approx \$1,220.94$$

$$\text{Tacoma: } \left(\frac{4.00 \text{ gal}}{100 \text{ mi}} \right) \left(\frac{12,000 \text{ mi}}{1} \right) \left(\frac{\$3.15}{\text{gal}} \right) = \$1,512.00$$

Avalon is lower by about \$291.06

For Love of the Math: Here's a great puzzle for some of you to attempt if interested. This one comes from the classic radio show, Car Talk, but we've mixed up some of the information so you won't just Google the answer!

A single father and his daughter have two vehicles: a gas guzzling SUV that gets 10 miles per gallon, and a sleek hybrid that gets 50 miles per gallon. The daughter can take the SUV in for a tune-up and increase the mileage to 12 miles per gallon. The father wants to buy a new hybrid that will get 150 miles per gallon. Only one option can be chosen, so the question is this: which option provides a greater improvement in miles per gallon for the household?

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