10.2 Plane Curves and Parametric Equations

Definition: If f & g are continuous functions of t on an interval I, then the equations

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

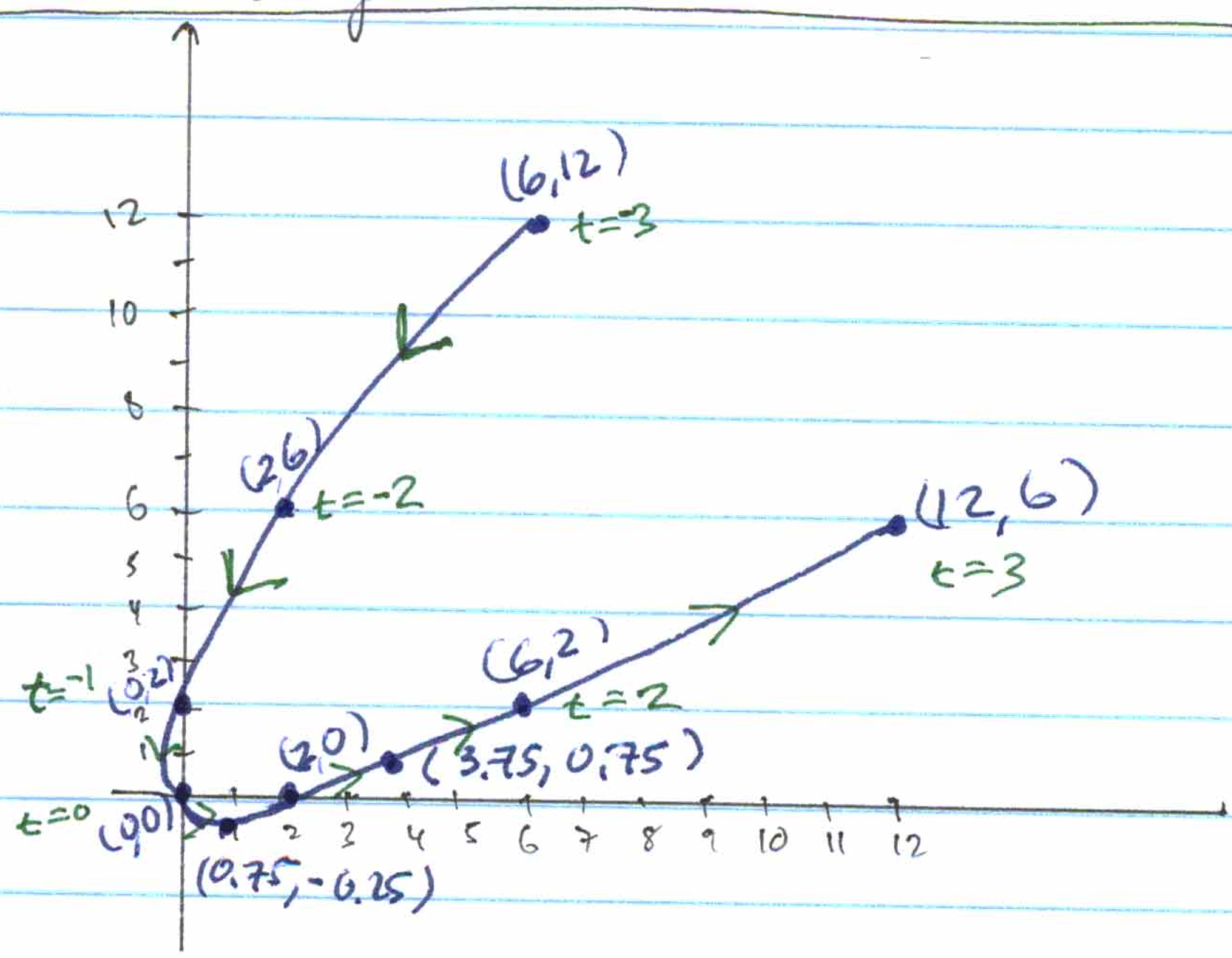
are called parametric equations and t is called the parameter. The set of points (x,y) obtained as t varies over I is the graph of the parametric equations.

Ex #8 Sketch the curve represented by the parametric equations and write the corresponding rectangular equation by eliminating the parameter.

$$\begin{cases} x = t^2 + t \\ y = t^2 - t \end{cases} \qquad t \in [-3, 3]$$

t	x	y
-3	6	12
-2	2	6
-1	0	2
0	0	0
0.5	0.75	-0.25
1	2	0
1.5	3.75	0.75
2	6	2
3	12	6

Just plug-in values for t and find x & y.



Think "t-clock"

$\boxed{10.2}$ #8 cont'd

It can be tricky to eliminate t and get an equation with only x & y, but let's try.

$$\begin{cases} x = t^2 + t \\ y = t^2 - t \end{cases}$$

solve for t!

$$x - y = (t^2+t) - (t^2-t)$$
$$x - y = t^2 + t - t^2 + t$$
$$x - y = 2t$$
$$\boxed{\frac{x-y}{2} = t} \longrightarrow \text{Substitute}$$

into either $x = t^2 + t$ or $y = t^2 - t$

$$x = t^2 + t$$
$$\boxed{x = \left(\frac{x-y}{2}\right)^2 + \left(\frac{x-y}{2}\right)} \quad \text{or} \quad \boxed{y = \left(\frac{x-y}{2}\right)^2 - \left(\frac{x-y}{2}\right)}$$
$$4 \cdot x = \left[\frac{(x-y)^2}{4} + \frac{x-y}{2}\right]\frac{4}{1}$$
$$4x = (x-y)^2 + 2(x-y)$$
$$-4x + 4x = x^2 - 2xy + y^2 + 2x - 2y - 4x$$
$$0 = x^2 - 2xy + y^2 - 2x - 2y$$

← If you want??

Restrictions / Range of values for x & y.

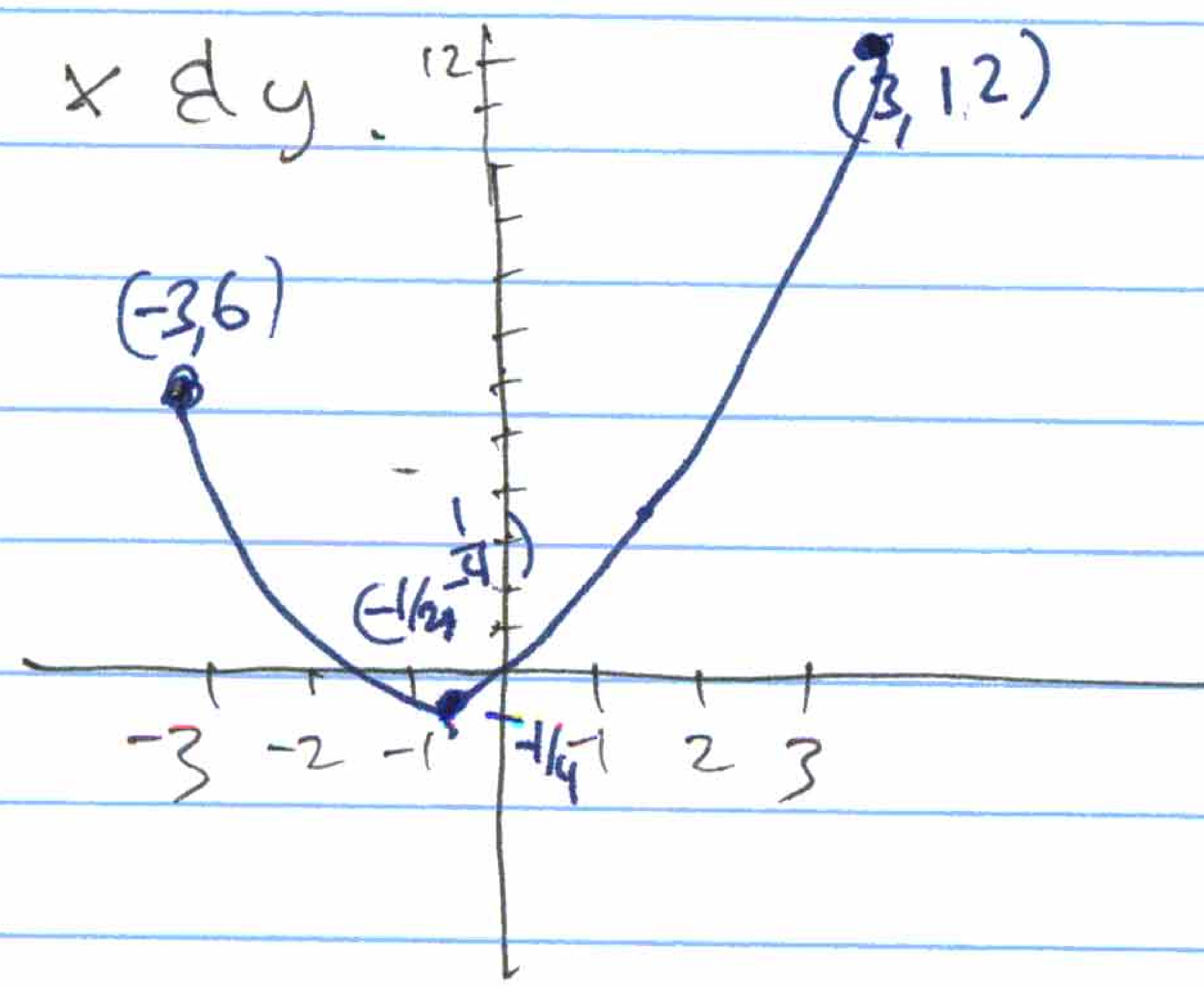
for $t \in [-3, 3]$ $\quad x = t^2 + t$

$$x'(t) = 2t + 1$$

minimum at $0 = 2t + 1$

$$-1 = 2t$$
$$-\tfrac{1}{2} = t \rightarrow x\left(-\tfrac{1}{2}\right) = -\tfrac{1}{4}$$

MAX at $(3, 12)$



$$x \in [-\tfrac{1}{4}, 12]$$

similarly, $\quad y \in [-\tfrac{1}{4}, 12]$

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Example: Use your graphing utility to graph

$$\begin{cases} x = t^2 \\ y = t \end{cases}$$

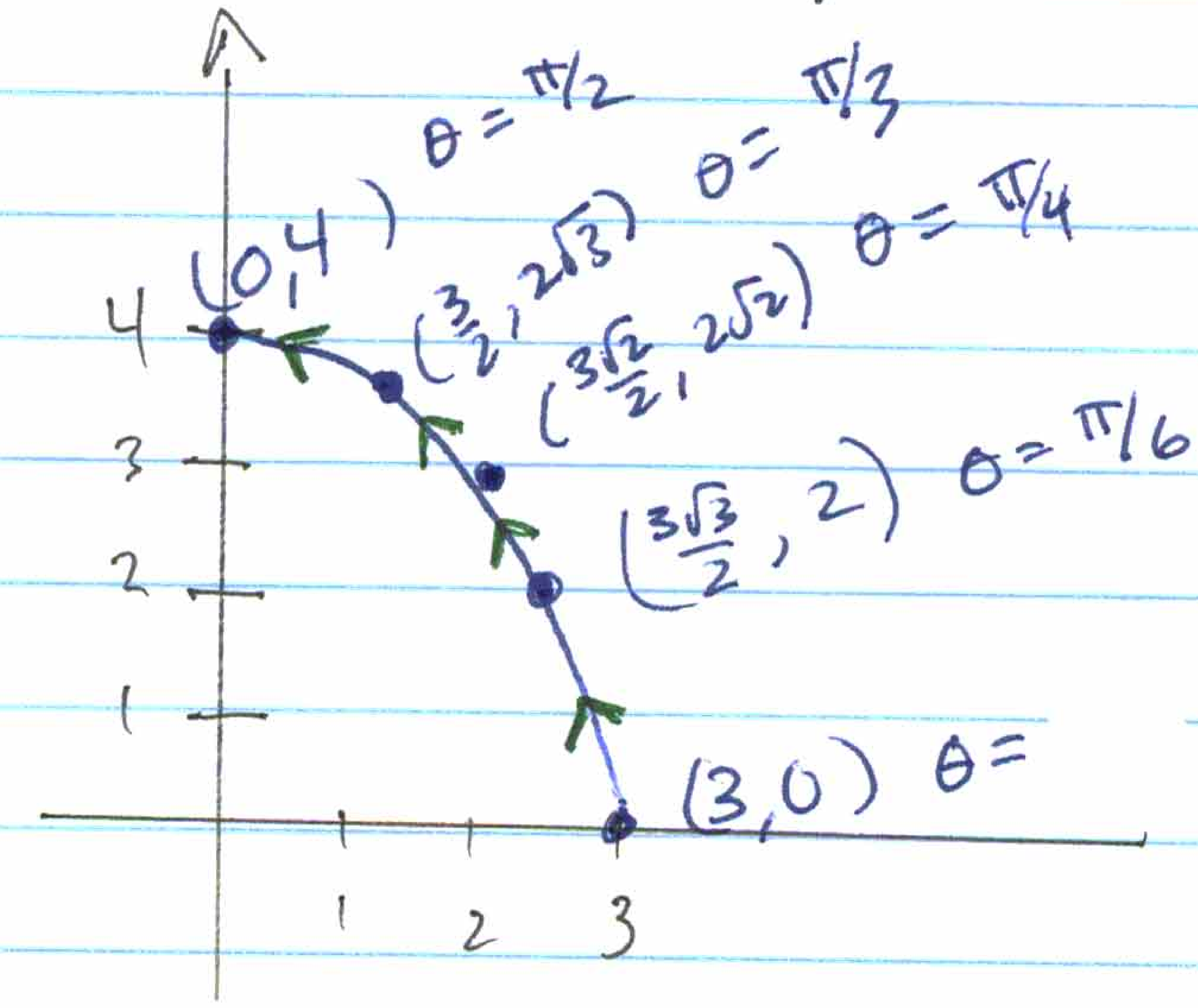
in parametric mode. $x = y^2$

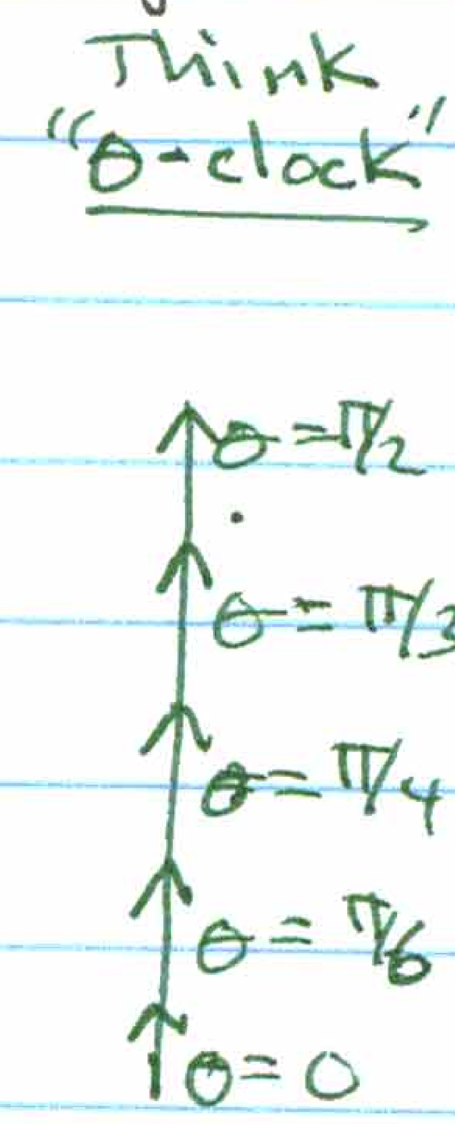
Example 3* sketch

$$\begin{cases} x(\theta) = 3\cos\theta \\ y(\theta) = 4\sin\theta \end{cases} \qquad 0 \le \theta \le \pi/2$$

then eliminate the parameter to find an equation in x & y.

θ	x	y
0	3	0
$\pi/6$	$\frac{3\sqrt{3}}{2}$	2
$\pi/4$	$\frac{3\sqrt{2}}{2}$	$2\sqrt{2}$
$\pi/3$	$\frac{3}{2}$	$2\sqrt{3}$
$\pi/2$	0	4





Eliminate θ:

$$x = 3\cos\theta \qquad \& \qquad y = 4\sin\theta$$

$$\frac{x}{3} = \cos\theta \qquad\qquad \frac{y}{4} = \sin\theta$$

$$\left(\frac{x}{3}\right)^2 = \cos^2\theta \qquad\qquad \left(\frac{y}{4}\right)^2 = \sin^2\theta$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\boxed{\frac{x^2}{9} + \frac{y^2}{16} = 1}$$

ellipse; center (0,0)

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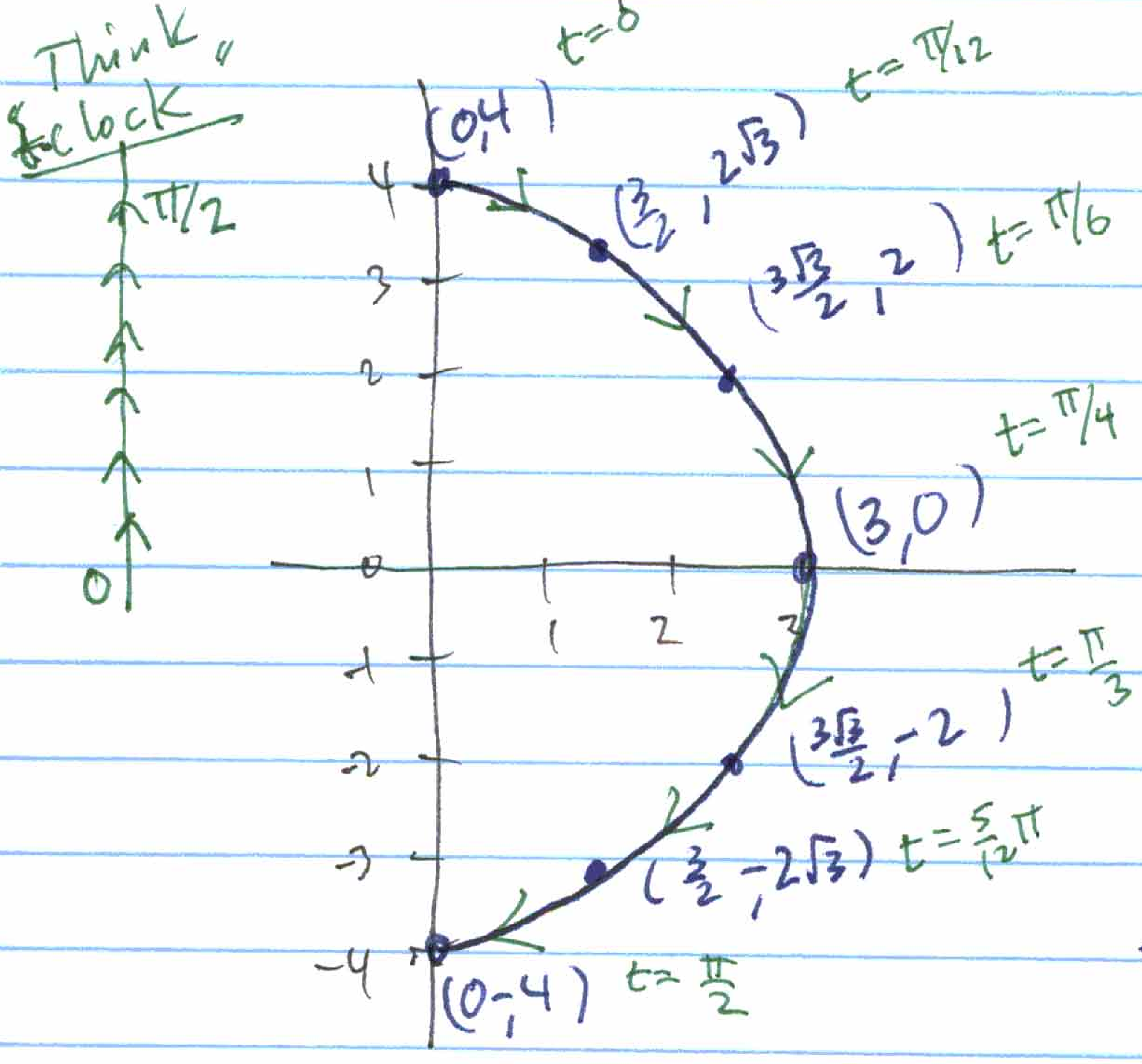
Let's "modify" this situation: $\begin{cases} x(t) = 3\sin(2t) \\ y(t) = 4\cos(2t) \end{cases}$ $t \in [0, \frac{\pi}{2}]$



t	x	y
0	0	4
$\frac{\pi}{12}$	$\frac{3}{2}$	$2\sqrt{3}$
$\frac{\pi}{6}$	$\frac{3\sqrt{3}}{2}$	2
$\frac{\pi}{4} = \frac{3\pi}{12}$	3	0
$\frac{\pi}{3} = \frac{4\pi}{12}$	$\frac{3\sqrt{3}}{2}$	-2
$\frac{5}{12}\pi$	$\frac{3}{2}$	$-2\sqrt{3}$
$\frac{\pi}{2} = \frac{6\pi}{12}$	0	-4

compare with

$$\boxed{\begin{cases} x(t) = 3\cos\left(\frac{\pi}{2} - 2t\right) \\ y(t) = 4\sin\left(\frac{\pi}{2} - 2t\right) \end{cases}}$$

$x = 3\sin(2t)$	$y = 4\cos(2t)$
$\frac{x}{3} = \sin(2t)$	$\frac{y}{4} = \cos(2t)$
$\left(\frac{x}{3}\right)^2 = \sin^2(2t)$	$\left(\frac{y}{4}\right)^2 = \cos^2(2t)$
$\frac{x^2}{9} = \sin^2(2t)$	$\frac{y^2}{16} = \cos^2(2t)$

$$\sin^2(2t) + \cos^2(2t) = 1$$

$$\boxed{\frac{x^2}{9} + \frac{y^2}{16} = 1}$$ SAME Ellipse

① we started at a different point
② we moved in the opposite direction
③ we moved "faster" ← twice as fast

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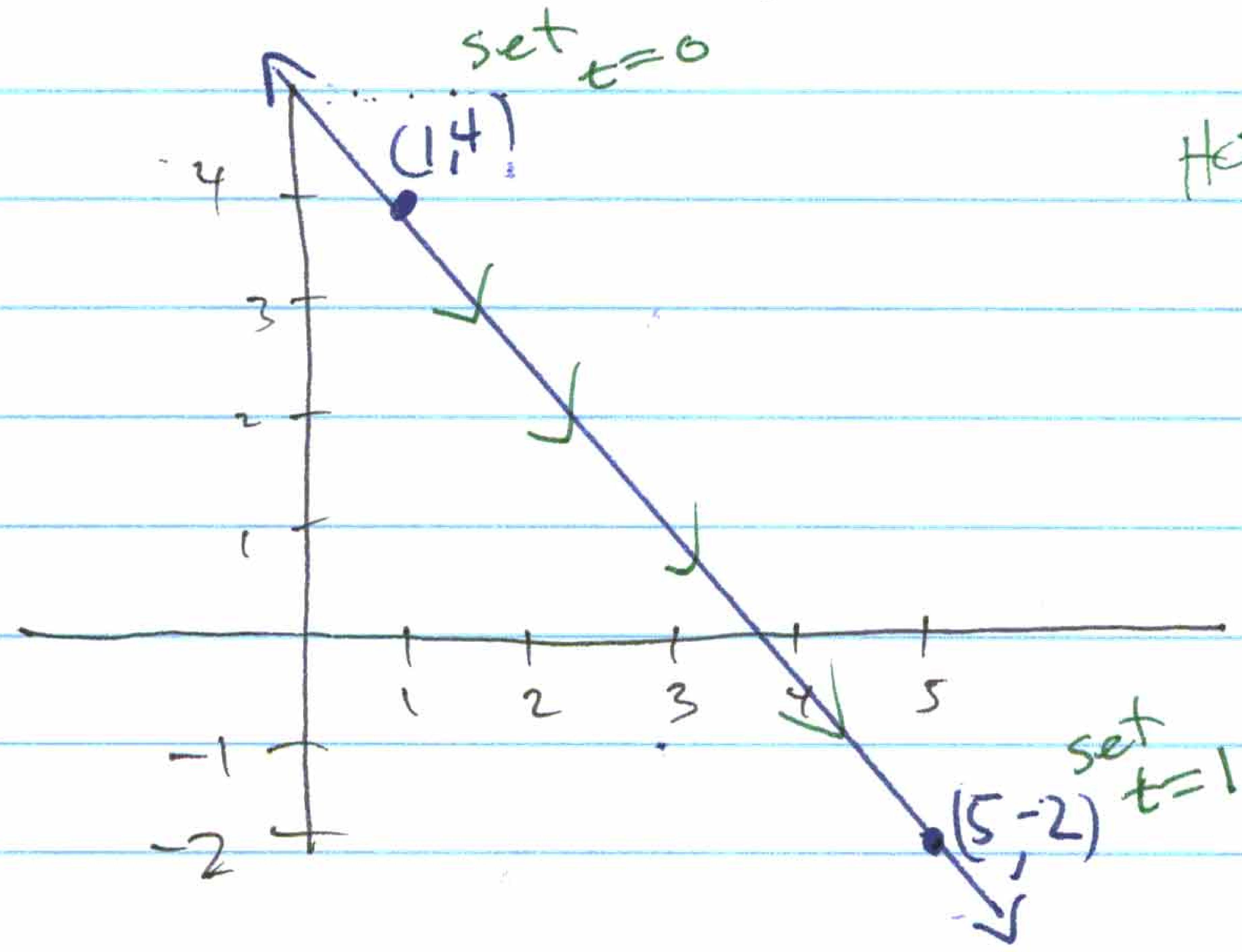
#44 Find a set of parametric equations for the line that passes through (1,4) and (5,-2).



How? $\left\{ \begin{array}{l} t=0 \rightarrow x=1 \ \& \ y=4 \\ t=1 \rightarrow x=5 \quad y=-2 \end{array} \right.$

$x = 1+4t$

$y = 4-6t$

Try on graphing calculator

$\begin{cases} x(t) = 1+4t \\ y(t) = 4-6t \end{cases}$

or $\begin{cases} x(t) = 1+2t \\ y(t) = 4-3t \end{cases}$ ← "slower"

or $\begin{cases} x(t) = 1+2t^2 \\ y(t) = 4-3t^2 \end{cases}$ Not uniform speed

$y - y_1 = m(x - x_1)$

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-2)-(4)}{(5)-1} = \frac{-6}{4} = \frac{-3}{2}$

$y - (4) = \frac{-3}{2}[x - (1)]$

$y - 4 = \frac{-3}{2}(x-1)$

$4 + y - 4 = -\frac{3}{2}x + \frac{3}{2} + \frac{8}{2}$

$\boxed{y = \frac{-3}{2}x + \frac{11}{2}}$

$x = 1+4t$

$x - 1 = 4t$

$\frac{x-1}{4} = t$

$y = 4-6t$

$y = 4 - 6\left(\frac{x-1}{4}\right)$

$y = 4 - \frac{3}{2}(x-1)$

$y = 4 - \frac{3}{2}x + \frac{3}{2}$

$y = \frac{8}{2} - \frac{3}{2}x + \frac{3}{2}$

$\boxed{y = \frac{-3}{2}x + \frac{11}{2}}$