10.3 Parametric Equations and Calculus

Definition: A curve C represented by $x = f(t)$ and $y = g(t)$ on an interval I is called smooth if $f'(t)$ and $g'(t)$ are continuous on I and not simultaneously 0, except possibly at the endpoints of I. The curve C is called piecewise smooth if it is smooth on each subinterval of some partition of I.

Theorem 10.7 - Parametric form of the Derivative

If a smooth curve C is given by the equations $x = f(t)$ and $y = g(t)$, then the slope of C at (x,y) is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \quad \frac{dx}{dt} \neq 0$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$$

← consider the notation very carefully!

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left[\frac{d^2y}{dx^2}\right] = \frac{\frac{d}{dt}\left[\frac{d^2y}{dx^2}\right]}{\frac{dx}{dt}}$$

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#2 Find $\frac{dy}{dx}$ for $\begin{cases} x(t) = \sqrt[3]{t} = t^{1/3} \\ y(t) = 4 - t \end{cases}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{-1}{\frac{1}{3}t^{-2/3}}$$

$$= -\frac{3t^{2/3}}{1}$$

$$\boxed{\frac{dy}{dx} = -3t^{2/3}}$$

$$\frac{dy}{dt} = \frac{d}{dt}[4-t]$$

$$\frac{dy}{dt} = 0 - 1 \quad , \frac{dy}{dt} = -1$$

$$\frac{dx}{dt} = \frac{d}{dt}[t^{1/3}]$$

$$\frac{dx}{dt} = \frac{1}{3}t^{-2/3}$$

#10 Find $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ at the given parameter value.

$\begin{cases} x(\theta) = \cos(\theta) \\ y(\theta) = 3\sin(\theta) \end{cases}$ at $\theta = 0$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{3\cos(\theta)}{-\sin(\theta)}$$

$$\frac{dy}{dx} = -3\cot(\theta)$$

$$\frac{dy}{d\theta} = 3\cos(\theta)$$

$$\frac{dx}{d\theta} = -\sin(\theta)$$

$$\left.\frac{dy}{dx}\right|_{\theta=0} = \frac{-3(1)}{0} = \text{undefined}$$

$\boxed{10.3}$ ~~#10 cont'd~~

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left[\frac{dy}{dx}\right]}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}\left[-3\cot(\theta)\right]}{-\sin(\theta)}$$

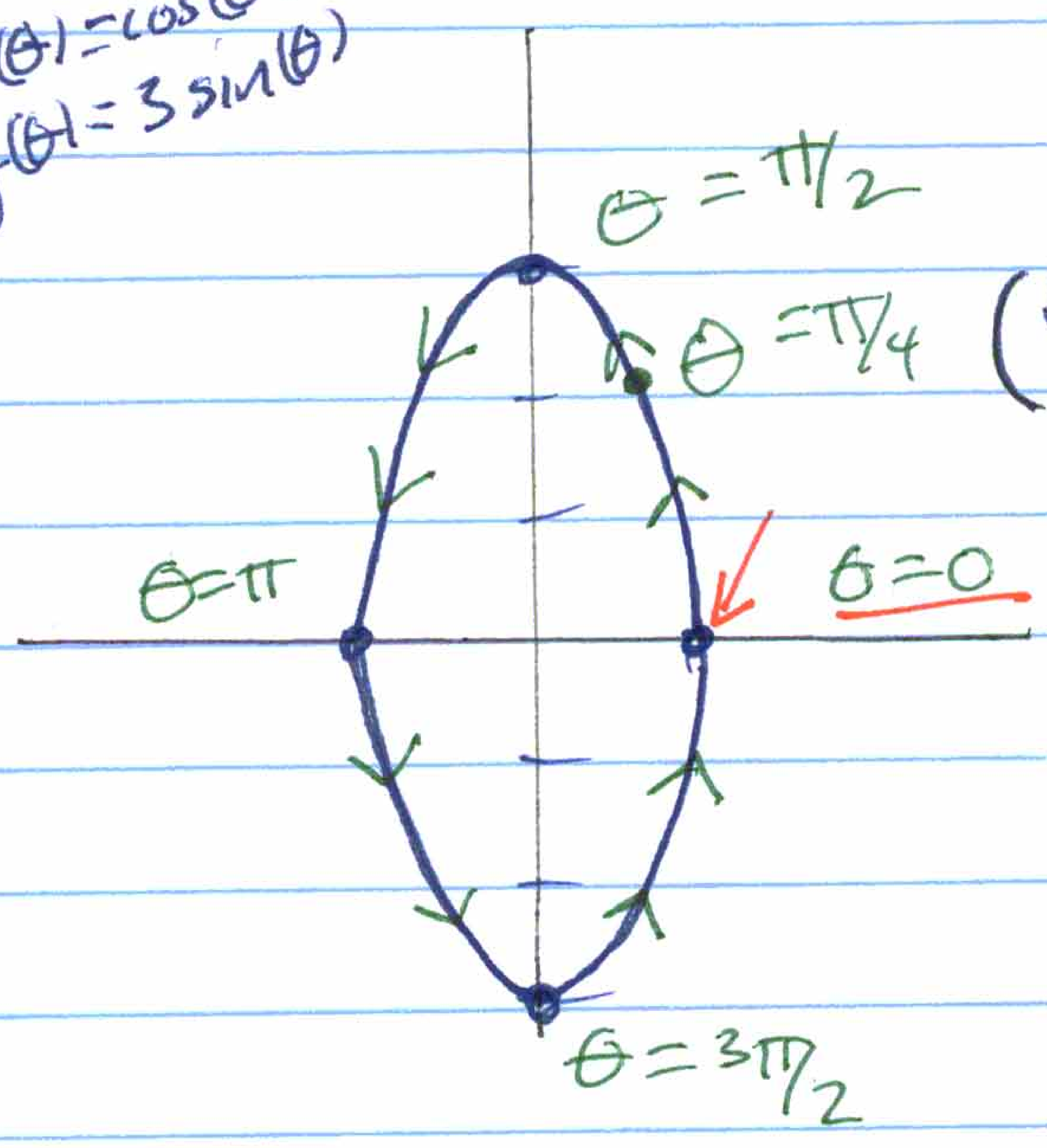
$$\frac{d^2y}{dx^2} = \frac{-3\frac{d}{d\theta}[\cot(\theta)]}{-\sin(\theta)}$$

$$\frac{d^2y}{dx^2} = \frac{-3(-\csc^2\theta)}{-\sin(\theta)}$$

$$\boxed{\frac{d^2y}{dx^2} = -3\csc^3(\theta)}$$

$$\left.\frac{d^2y}{dx^2}\right|_{\theta=0} = -3\csc^3(0) = \text{undefined}$$

$\begin{cases} x(\theta) = \cos(\theta) \\ y(\theta) = 3\sin(\theta) \end{cases}$



#10*** Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $\theta = \pi/4$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/4} = \frac{-3\cos(\pi/4)}{\sin(\pi/4)}$$

$\boxed{10.3}$ #10** cont'd

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/4} = \frac{-3\left(\frac{\sqrt{2}}{2}\right)}{\left(\frac{\sqrt{2}}{2}\right)}$$

$$\boxed{\left.\frac{dy}{dx}\right|_{\theta=\pi/4} = -3}$$

$$\left.\frac{d^2y}{dx^2}\right|_{\theta=\pi/4} = -3\csc^3(\pi/4)$$

$$= -3\left(\sqrt{2}\right)^3$$

$$= -3\left(2\sqrt{2}\right)$$

$$\boxed{\left.\frac{d^2y}{dx^2}\right|_{\theta=\pi/4} = -6\sqrt{2}}$$

Example: Find the equation of the tangent line to the curve (C) defined by the equation $\frac{x^2}{9}+\frac{y^2}{16}=1$, at the poin $M\left(\frac{-3\sqrt{2}}{2}, y\right)$, $y>0$.

We know $x_0 = \frac{3\sqrt{2}}{2}$ at M, let's find y_0 at M.

$$\frac{\left(\frac{-3\sqrt{2}}{2}\right)^2}{9} + \frac{y^2}{16} = 1$$

$$\frac{\frac{9\cdot 2}{4}}{9} + \frac{y^2}{16} = 1$$

$$-\frac{1}{2} + \frac{1}{2} + \frac{y^2}{16} = 1 - \frac{1}{2}$$

$$16\cdot\frac{y^2}{16} = \frac{1}{2}16$$

$\Rightarrow$ $y^2 = 8$

$\boxed{y_0 = 2\sqrt{2}}$

since $y>0$ at M.

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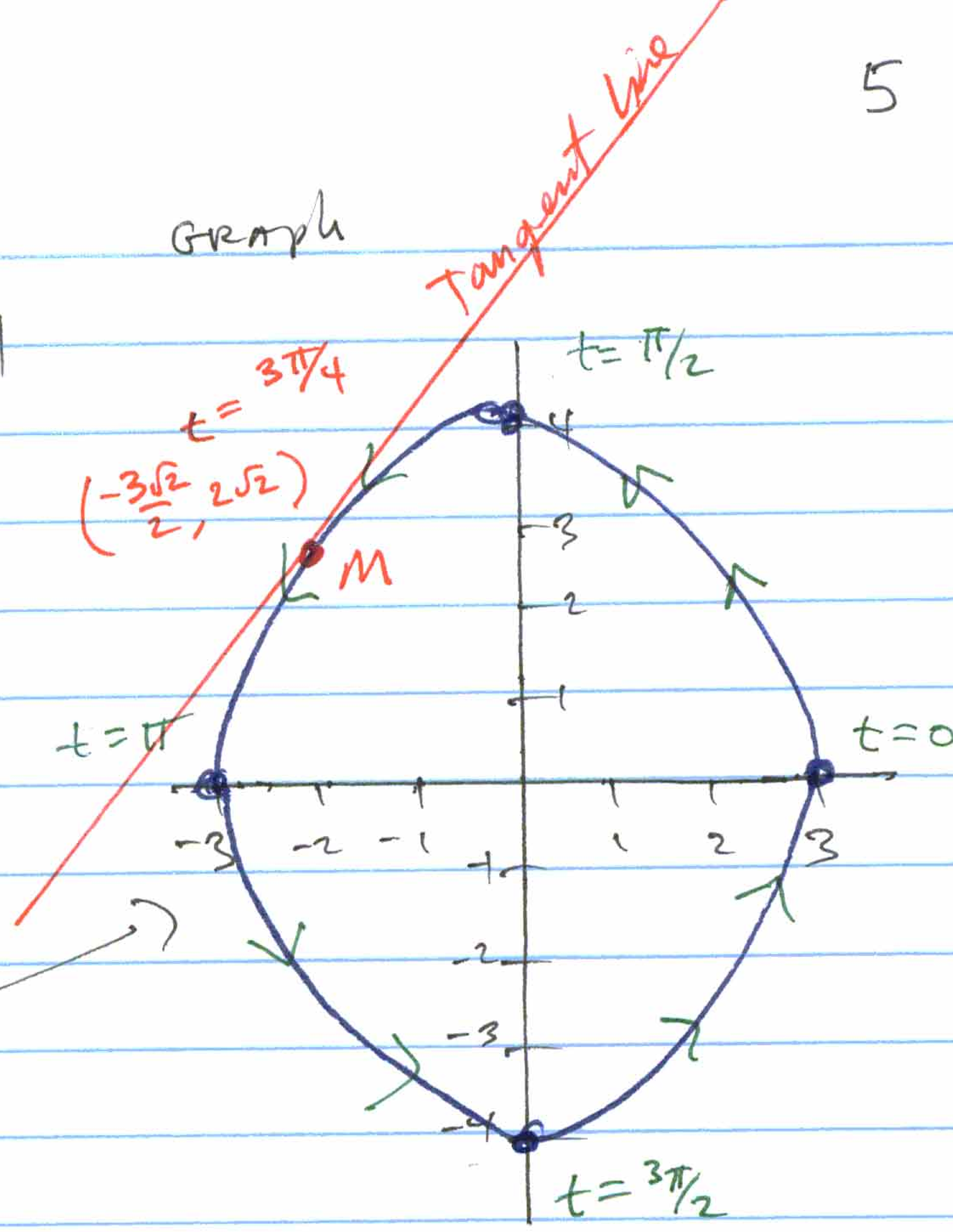
Example cont'd

$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

Graph



we can parametrize this curve:

$$\frac{x^2}{9} = \cos^2 t \qquad \sin^2 t = \frac{y^2}{16}$$

$$\frac{x}{3} = \cos(t) \qquad \sin(t) = \frac{y}{4}$$

$$x = 3\cos t \qquad 4\sin t = y$$

$$\boxed{\begin{cases} x(t) = 3\cos(t) \\ y(t) = 4\sin(t) \end{cases}}$$

Find t at M. $M = \left(\overset{x_0}{-\frac{3\sqrt{2}}{2}}, \overset{y_0}{2\sqrt{2}}\right)$

Solve: $3\cos(t) = -\frac{3\sqrt{2}}{2}$ & $2\sqrt{2} = 4\sin(t)$

$$\frac{1}{3}\cdot 3\cos(t) = \frac{1}{3}\left(-\frac{3\sqrt{2}}{2}\right) \qquad \frac{1}{4}\cdot 2\sqrt{2} = \frac{1}{4}\cdot 4\sin(t)$$

$$\cos(t) = -\frac{\sqrt{2}}{2} \qquad \frac{\sqrt{2}}{2} = \sin(t)$$

$$\boxed{t = \frac{3\pi}{4}}$$

To find the slope of the tangent line at $t = \frac{3\pi}{4}$, we need to find $\left.\frac{dy}{dx}\right|_{t=\frac{3\pi}{4}}$. So, find $\frac{dy}{dt}$ & $\frac{dx}{dt}$.

$$\frac{dy}{dt} = \frac{d}{dt}(4\sin(t)) \qquad \frac{dx}{dt} = \frac{d}{dt}(3\cos(t))$$

$$\frac{dy}{dt} = 4\cos(t) \qquad \frac{dx}{dt} = -3\sin(t)$$

$\boxed{10.3}$ Example 1 cont'd

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4\cos t}{-3\sin t}$$

$$\left.\frac{dy}{dx}\right|_{t=\frac{3\pi}{4}} = \frac{4\cos\left(\frac{3\pi}{4}\right)}{-3\sin\left(\frac{3\pi}{4}\right)}$$

$$M_{TANGENT} = \frac{4\left(-\frac{\sqrt{2}}{2}\right)}{-3\left(\frac{\sqrt{2}}{2}\right)}$$

$$\boxed{M_{TANGENT} = \frac{4}{3}}$$

Equation of Tangent at M

$$y - y_1 = m_{TAN}(x - x_1)$$

$$y - (2\sqrt{2}) = \left(\frac{4}{3}\right)\left[x - \left(-\frac{3\sqrt{2}}{2}\right)\right]$$

$$y - 2\sqrt{2} = \frac{4}{3}\left(x + \frac{3\sqrt{2}}{2}\right)$$

$$y - 2\sqrt{2} = \frac{4}{3}x + 2\sqrt{2}$$

$$\boxed{y = \frac{4}{3}x + 4\sqrt{2}}$$

can be parameterized as

$$\begin{cases} x(t) = t \\ y(t) = \frac{4}{3}t + 4\sqrt{2} \end{cases}$$

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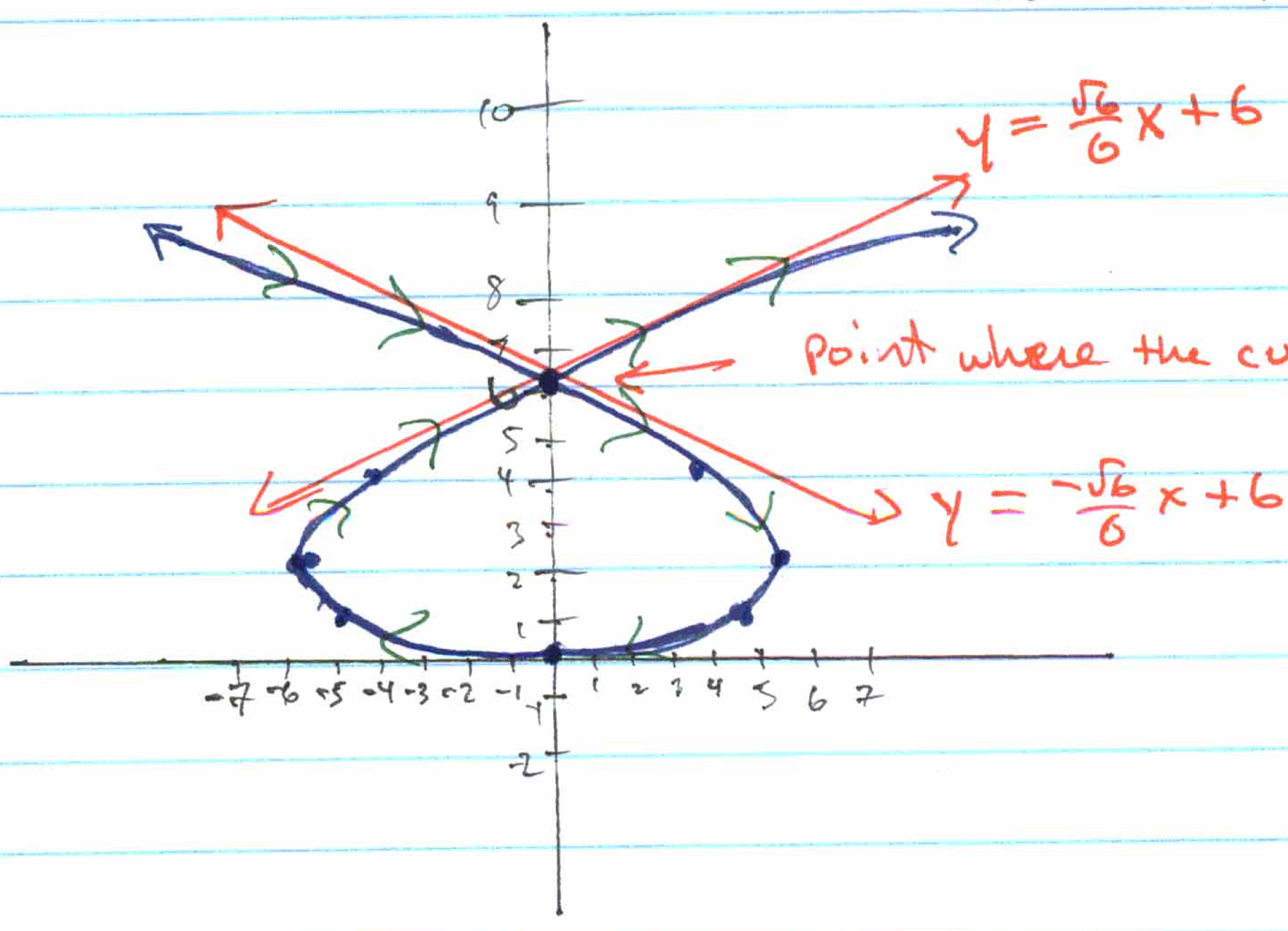
$$\begin{cases} x(t) = t^3 - 6t \\ y(t) = t^2 \end{cases}$$

Find the equations of the tangent lines at the point where the curve crosses itself



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#24 cont'd

Let's label with t & z the two different times when the curve arrives at the same point:

$$\begin{cases} t^3-6t = z^3-6z \\ t^2 = z^2 \end{cases}$$

From the lower equation, we know that $t^2=z^2$ when $z=\pm t$. Since z and t are different, we know z cannot be t. Therefore $\underline{z=-t}$. Plug this into the first equation:

$$t^3-6t=z^3-6z$$

Solve for t →

$$t^3-6t=(-t)^3-6(-t)$$

$$t^3-6t=-t^3+6t$$

$$-6t+t^3+t^3-6t=-t^3+6t+(-6t+t^3)$$

$$2t^3-12t=0$$

$$2t(t^2-6)=0$$

Either

$2t=0$, or $t^2-6=0$

$\boxed{t=0}$ $\quad t^2=6$

$\boxed{t=\pm\sqrt{6}}$

Since $t=0$ will make $z=0$, we cannot have this. This means that $t=\sqrt{6}$ and $z=-\sqrt{6}$. These are the two "times" when the curve crosses itself.

Now, we need to find $\left.\frac{dy}{dx}\right|_{t=\sqrt{6}}$ & $\left.\frac{dy}{dx}\right|_{t=-\sqrt{6}}$

$\boxed{10.3}$ ~~#24~~ cont'd

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{dy}{dt} = \frac{d}{dt}\left[t^2\right]$$

$$\boxed{\frac{dy}{dt} = 2t}$$

$$\frac{dx}{dt} = \frac{d}{dt}\left[t^3 - 6t\right]$$

$$\boxed{\frac{dx}{dt} = 3t^2 - 6}$$

$$\boxed{\frac{dy}{dx} = \frac{2t}{3t^2 - 6}}$$

$$\left.\frac{dy}{dx}\right|_{t=\sqrt{6}} = \frac{2(\sqrt{6})}{3(\sqrt{6})^2 - 6} = \frac{2\sqrt{6}}{3(6) - 6} = \frac{2\sqrt{6}}{12}$$

$$\boxed{m_{TAN_{t=\sqrt{6}}} = \frac{\sqrt{6}}{6}}$$

$$\left.\frac{dy}{dx}\right|_{t=-\sqrt{6}} = \frac{2(-\sqrt{6})}{3(-\sqrt{6})^2 - 6} = \frac{-2\sqrt{6}}{3(6) - 6} = \frac{-2\sqrt{6}}{12}$$

$$\boxed{m_{TAN_{t=-\sqrt{6}}} = \frac{-\sqrt{6}}{6}}$$

Find (x_1, y_1) when $t = \sqrt{6}$

$$\begin{cases} x(\sqrt{6}) = (\sqrt{6})^3 - 6(\sqrt{6}) = 0 \\ y(\sqrt{6}) = (\sqrt{6})^2 = 6 \end{cases}$$

$$(x_1, y_1) = (0, 6)$$

Equation of tangent at $t = \sqrt{6}$

$$y - y_1 = m_{TAN}(x - x_1)$$
$$y - (6) = \frac{\sqrt{6}}{6}(x - (0))$$
$$y - 6 = \frac{\sqrt{6}}{6}x$$
$$\boxed{y = \frac{\sqrt{6}}{6}x + 6}$$

$$\begin{cases} x(t) = t \\ y(t) = \frac{\sqrt{6}}{6}t + 6 \end{cases}$$

Equation of tangent at $t = -\sqrt{6}$

$$y - y_1 = m_{TAN}(x - x_1)$$
$$y - (6) = \frac{-\sqrt{6}}{6}(x - (0))$$
$$y - 6 = \frac{-\sqrt{6}}{6}x$$
$$\boxed{y = \frac{-\sqrt{6}}{6}x + 6}$$

$$\begin{cases} x(t) = t \\ y(t) = \frac{-\sqrt{6}}{6}t + 6 \end{cases}$$

If $\frac{dy}{dt} = 0$ & $\frac{dx}{dt} \neq 0$, at $t = t_0$, $\rightarrow$ Horizontal tangent at $(f(t_0), g(t_0))$

If $\frac{dy}{dt} \neq 0$ & $\frac{dx}{dt} = 0$, at $t = t_0$ $\rightarrow$ vertical tangent at $(f(t_0), g(t_0))$.

$\boxed{10.3}$

#26 Find the points of horizontal and vertical tangency to

$$\begin{cases} x(t) = 2t \\ y(t) = 2(1-\cos(t)) \end{cases}, \quad -1 \leq t \leq 2\pi$$

$$\frac{dx}{dt} = \frac{d}{dt}[2t]$$

$$\frac{dx}{dt} = 2$$

$$\frac{dx}{dt} \neq 0$$

No Points of Vertical Tangency

$$\frac{dy}{dt} = \frac{d}{dt}[2-2\cos t]$$

$$= 0 - 2(-\sin t)$$

$$\frac{dy}{dt} = 2\sin(t)$$

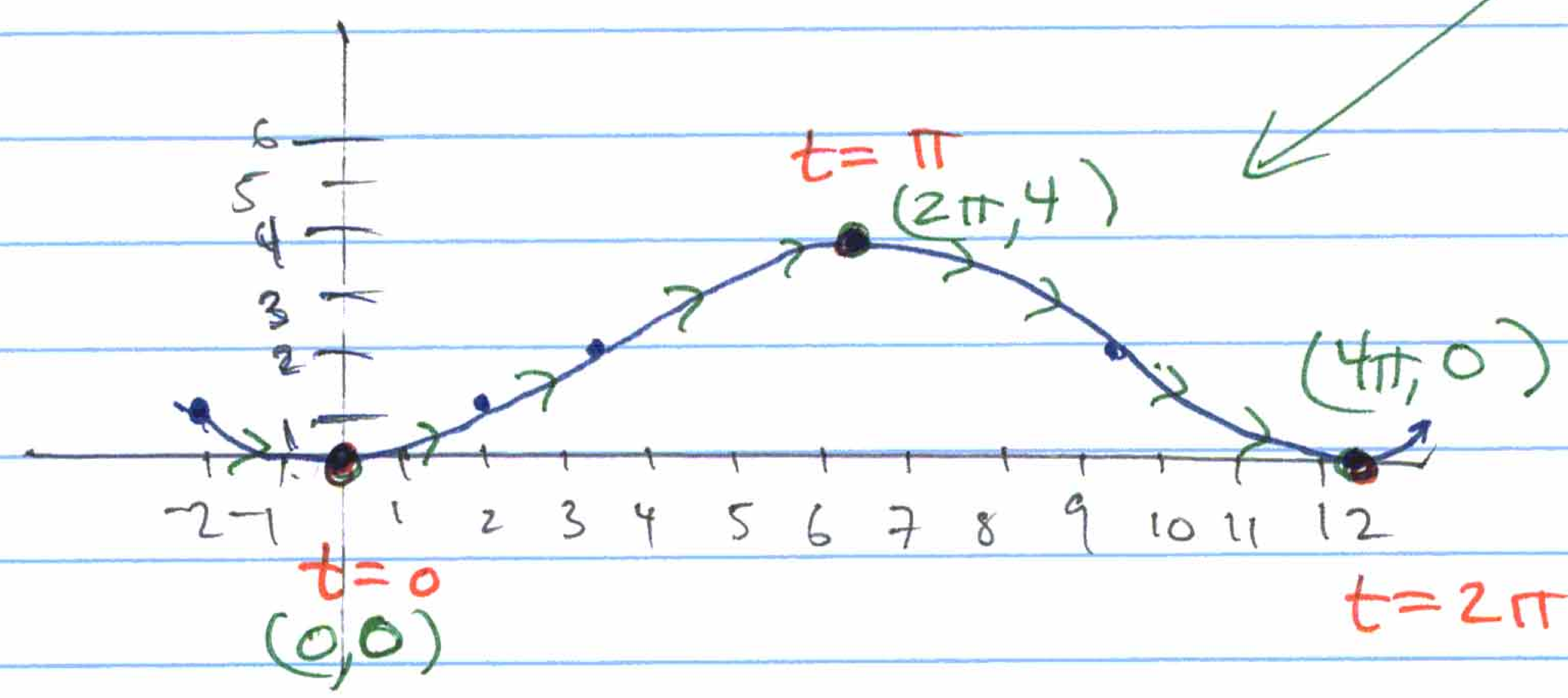
$$\frac{dy}{dt} = 0 \rightarrow 2\sin(t) = 0$$

$$\frac{1}{2} \cdot 2\sin(t) = \frac{1}{2} \cdot 0$$

$$\sin(t) = 0$$

$t = 0$, $t = \pi$, or $t = 2\pi$

3 points of Horizontal Tangency



$$\begin{cases} x(0) = 2(0) = 0 \\ y(0) = 2(1-\cos(0)) \; 0 \end{cases}$$

$$\begin{cases} x(\pi) = 2(\pi) = 2\pi \\ y(\pi) = 2[1-\cos(\pi)] = 2[1+1] = 4 \end{cases}$$

$$\begin{cases} x(2\pi) = 2(2\pi) = 4\pi \\ y(2\pi) = 2(1-\cos(2\pi)) = 2 \cdot 0 = 0 \end{cases}$$

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#39 $\begin{cases} x(t) = 2t + \ln(t) \\ y(t) = 2t - \ln(t) \end{cases}$

Determine the t-intervals on which the curve is concave downward or concave upward.

$\frac{d^2y}{dx^2} > 0$ & $\frac{d^2y}{dx^2} < 0$

concave up Concave down

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2 - \frac{1}{t}}{2 + \frac{1}{t}}$$

$$\frac{dy}{dx} = \left(\frac{2 - \frac{1}{t}}{2 - \frac{1}{t}}\right)\left(\frac{\frac{t}{1}}{\frac{t}{1}}\right)$$

$$\boxed{\frac{dy}{dx} = \frac{2t - 1}{2t + 1}}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left[\frac{2t-1}{2t+1}\right]}{\left(2 + \frac{1}{t}\right)}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{(2t+1)\frac{d}{dt}(2t-1) - (2t-1)\frac{d}{dt}(2t+1)}{(2t+1)^2}}{2 + \frac{1}{t}}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{(2t+1)(2) - (2t-1)(2)}{(2t+1)^2}}{\frac{2t+1}{t}}$$

$$\frac{d^2y}{dx^2} = \left[\frac{4t + 2 - 4t + 2}{(2t+1)^2}\right] \cdot \left[\frac{t}{2t+1}\right]$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{4t}{(2t+1)^3}}$$

$\boxed{10.3}$ #39 cont'd

Solve: $\frac{4t}{(2t+1)^3} = 0$

$4t = 0$

$t = 0$

$(2t+1)^3 = 0$

$2t+1=0$

$2t=-1$

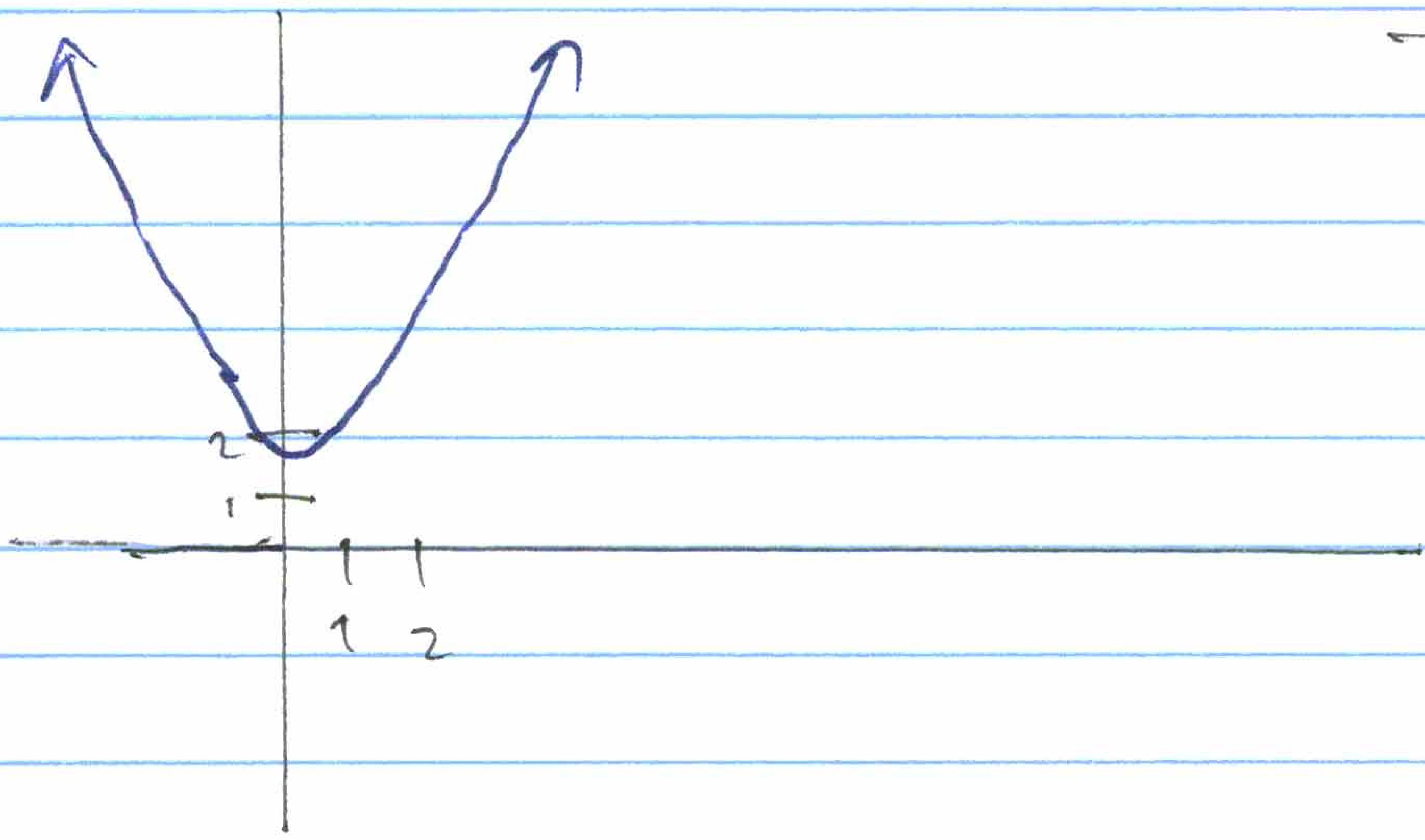
$t=-1/2$

$\frac{d^2y}{dx^2}$: ln(t) is undefined | ln(t) is undefined | + + + + + +

Test $t=-1$ | $t=-1/2$ | $t=0$

concave up on $(0,\infty)$

the curve is concave up.



From $\boxed{7.4}$ – Arc Length, $y=f(x)$ on $[a,b]$

↑ smooth curve on $[a,b]$

$$s = \text{Arc length between } a \,\&\, b = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

Theorem $\boxed{10.8}$ Arc Length in Parametric Form

If a smooth curve C is given by $x=f(t)$ and $y=g(t)$ such that C does not intersect itself on the interval $a \le t \le b$ (except possibly at the endpoints); then the arclength of C over the interval is given by

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt$$

$\boxed{10.3}$

#44 $\begin{cases} x(t) = \ln t \\ y(t) = t+1 \end{cases}$

Write an integral that represents the arclength over $1 \le t \le 6$.

$$S = \int_a^b \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\, dt$$

$$S = \int_1^6 \sqrt{\left(\frac{1}{t}\right)^2 + (1)^2}\, dt$$

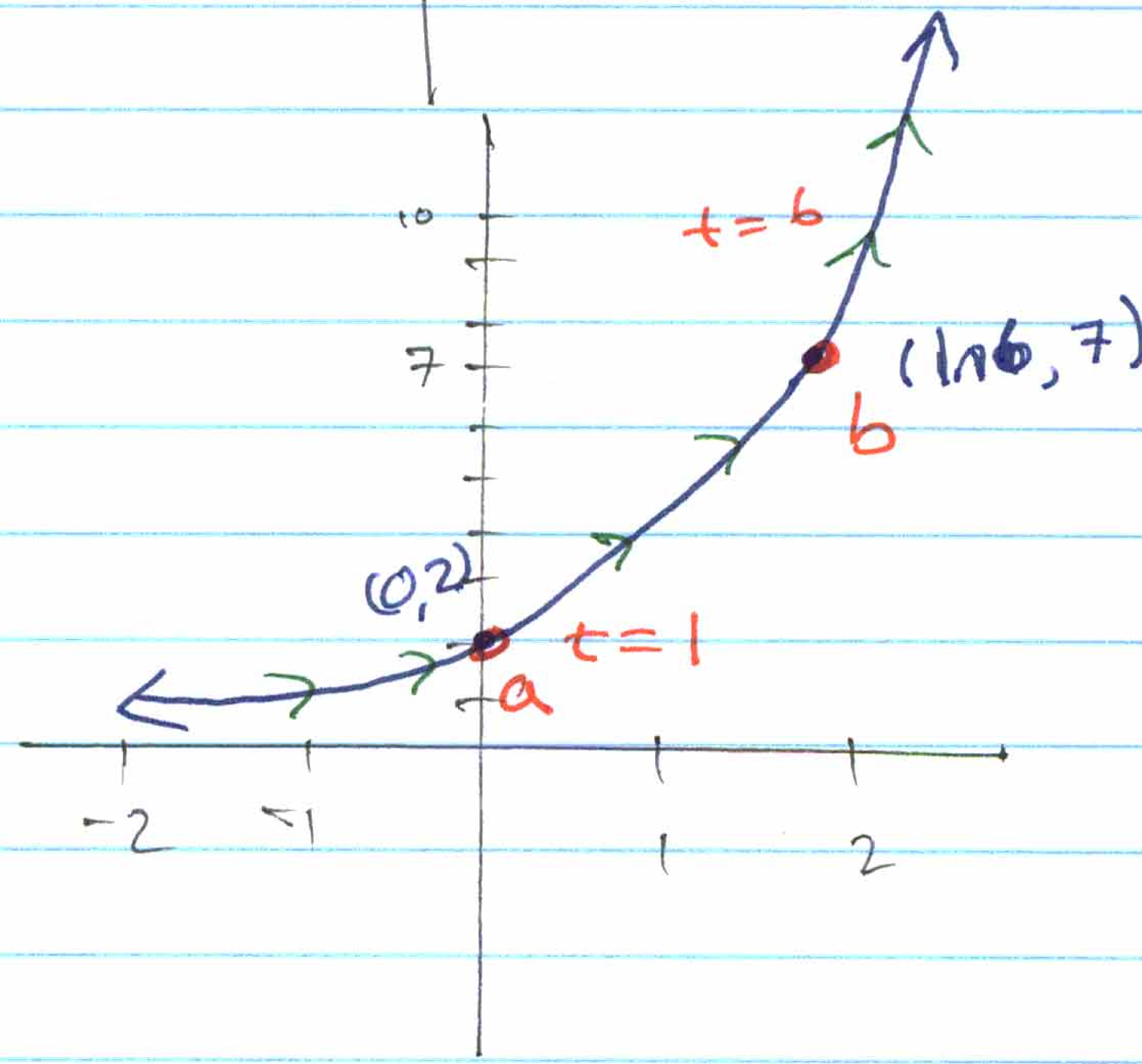
$$S = \int_1^6 \sqrt{\frac{1}{t^2} + 1}\, dt$$

$$S = \int_1^6 \sqrt{\frac{1+t^2}{t^2}}\, dt$$

$$\boxed{S = \int_1^6 \frac{\sqrt{1+t^2}}{t}\, dt}$$

$\frac{dx}{dt} = \frac{1}{t}$

$\frac{dy}{dt} = 1$



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Ex. Find the circumfeume of a circel with radius a.

$$\begin{cases} x(t) = a\cos(t) \\ y(t) = a\sin(t) \end{cases} \qquad 0 \le t \le 2\pi$$

$$\frac{dx}{dt} = \frac{d}{dt}[a\cos(t)] = -a\sin(t)$$

$$\frac{dy}{dt} = \frac{d}{dt}[a\sin(t)] = a\cos(t)$$

$$s = \int_0^{2\pi} \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\, dt$$

$$s = \int_0^{2\pi} \sqrt{[-a\sin(t)]^2 + [a\cos(t)]^2}\, dt$$

$$s = \int_0^{2\pi} \sqrt{a^2\sin^2(t) + a^2\cos^2(t)}\, dt$$

$\boxed{10.3}$

#55 Find the arclength of the curve $\begin{cases} x(t) = a(t - \sin t) \\ y(t) = a(1 - \cos t) \end{cases}$
on $0 \leq t \leq 2\pi$