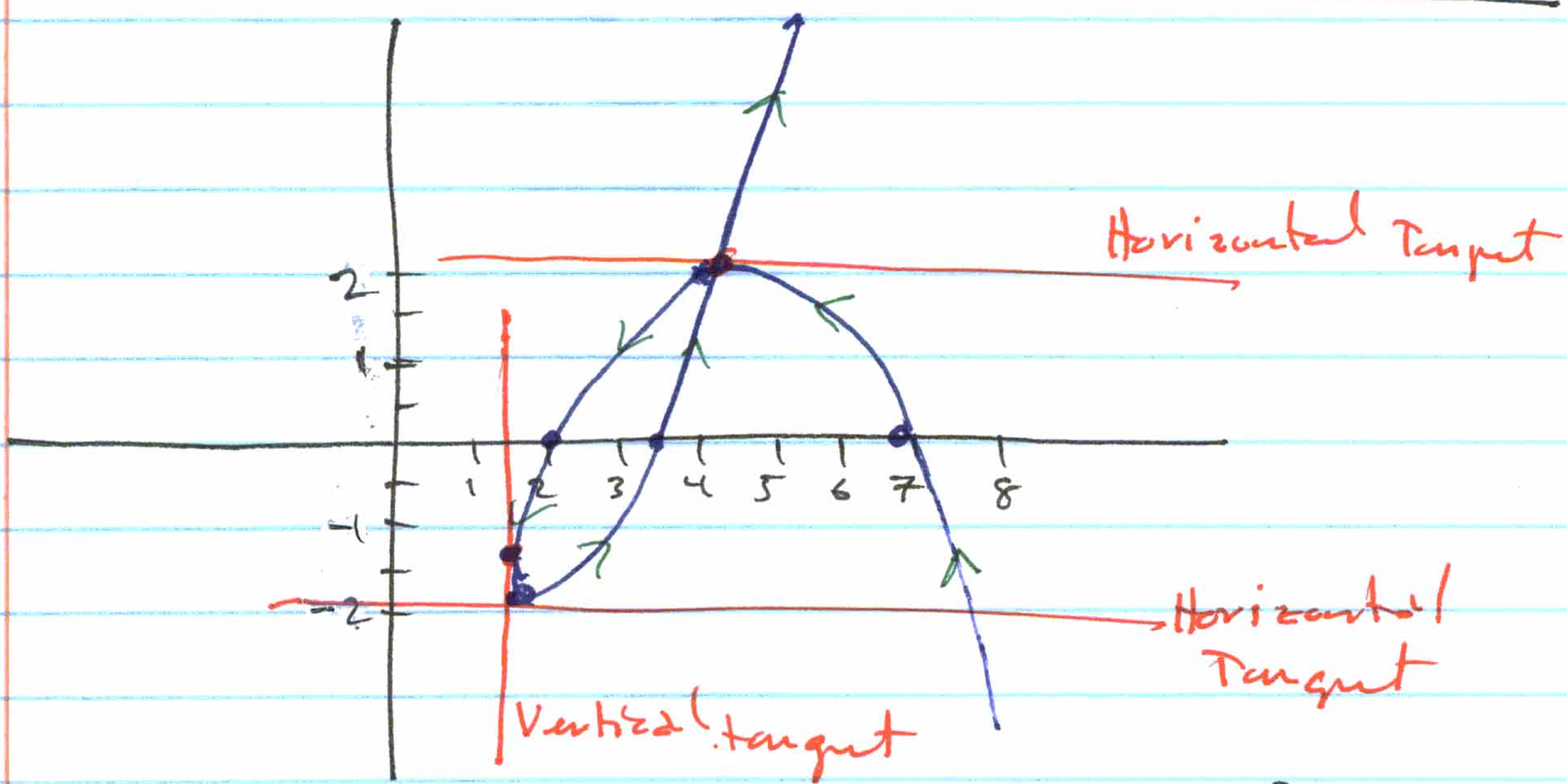
10.3 Examples –

#30 Find all points of horizontal and vertical tangency to the curve

$$\begin{cases} x(t) = t^2 - t + 2 \\ y(t) = t^3 - 3t \end{cases}$$



$$\frac{dx}{dt} = \frac{d}{dt}\left[t^2 - t + 2\right]$$

$$\frac{dx}{dt} = 2t - 1$$

$$\frac{dy}{dt} = \frac{d}{dt}\left[t^3 - 3t\right]$$

$$\frac{dy}{dt} = 3t^2 - 3$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t - 1}$$

$$\frac{dy}{dt} = 0$$

$$3t^2 - 3 = 0$$

$$3t^2 = 3$$

$$t^2 = 1$$

$$t = 1 \text{ or } t = -1$$

$$\frac{dx}{dt} = 0$$

$$2t - 1 = 0$$

$$t = 1/2$$

10.3

#30 cont'd

$$\frac{dy}{dx} = 0 \quad \text{if} \quad \frac{dy}{dt} = 0 \text{ and } \frac{dx}{dt} \neq 0$$

$$\frac{dy}{dx} = \text{undefined if} \quad \frac{dx}{dt} = 0 \text{ and } \frac{dy}{dt} \neq 0$$

Horizontal tangents: $t = 1$ or $t = -1$

$$\left.\frac{dy}{dx}\right|_{t=1} = \frac{3(1)^2 - 3}{2(1) - 1} = \frac{0}{1} = 0$$

$$\left.\frac{dy}{dx}\right|_{t=-1} = \frac{3(-1)^2 - 3}{2(-1) - 1} = \frac{3-3}{-2-1} = \frac{0}{-3} = 0$$

Point: $(x(1), y(1)) = (2, -2)$

$(x(-1), y(-1)) = (4, 2)$

Vertical Tangent: $t = \frac{1}{2}$

$$\left.\frac{dy}{dx}\right|_{t=1/2} = \frac{3(\frac{1}{2})^2 - 3}{2(\frac{1}{2}) - 1} = \frac{\frac{3}{4} - 3}{1-1} = \frac{-9/4}{0} = \text{undefined}$$

Point: $(x(\frac{1}{2}), y(\frac{1}{2})) = (\frac{7}{4}, \frac{-11}{8})$

10.3

#48: Find the arclength of the curve

$$\begin{cases} x(t) = t^2+1 \\ y(t) = 4t^3+3 \end{cases} \quad \text{over} \quad -1 \le t \le 0.$$

$$\frac{dx}{dt} = \frac{d}{dt}(t^2+1) \qquad \frac{dy}{dt} = \frac{d}{dt}[4t^3+3]$$

$$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = 12t^2$$

$$\text{arclength} = s = \int_{-1}^{0} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left[\frac{dy}{dt}\right]^2}\, dt$$

$$s = \int_{-1}^{0} \sqrt{(2t)^2 + (12t^2)^2}\, dt$$

$$s = \int_{-1}^{0} \sqrt{4t^2 + 144t^4}\, dt$$

$$s = \int_{-1}^{0} \sqrt{4t^2}\cdot\sqrt{(1+36t^2)}\, dt$$

$$s = \int_{-1}^{0} (-2t)\sqrt{1+36t^2}\, dt$$

Look!

$\sqrt{4t^2} = -2t$ since $-1 \le t \le 0$

$$s = -\int_{u=37}^{u=1} 2t\,(u)^{1/2}\cdot\left(\frac{du}{72t}\right)$$

$$s = \frac{-1}{36}\int_{37}^{1} u^{1/2}\, du$$

$$s = \frac{-1}{36}\left[\frac{2}{3}u^{3/2}\right]_{37}^{1}$$

Let $u = 1+36t^2$

$\frac{du}{dt} = 72t$

$\frac{du}{72t} = dt$

$t=0$

$u = 1+36(0)^2$

$u = 1$

$t=-1$

$u = 1+36(-1)^2$

$u = 37$

10.3

#48 cont'd

$$S = -\frac{1}{54}\left[u^{3/2}\right]_{37}^{1}$$

$$S = -\frac{1}{54}\left[(1)^{3/2} - (37)^{3/2}\right]$$

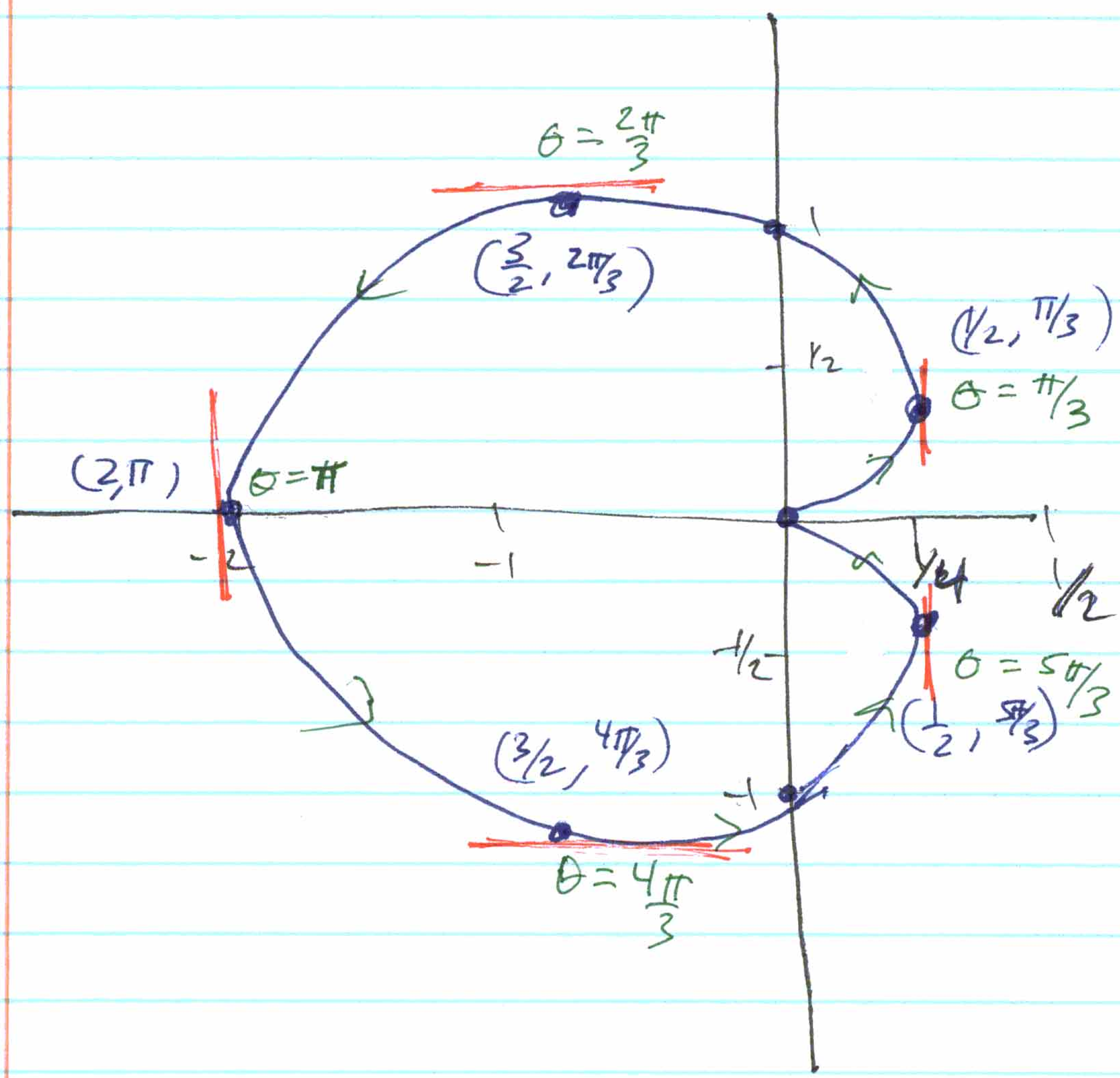
$$S = -\frac{1}{54}\left[1 - 37\sqrt{37}\right] \approx \underline{4.149}$$

10.4

Examples –

Find all points on the cardioid $r = 1 - \cos(\theta)$ at which there is a horizontal tangent line, a vertical tangent line, or a "singular point"

– "singular point" is where dx/dt & $\frac{dy}{dt}$ are both zero. No general statement can be made about parametric curves at singular points; they must be analyzed case by case:



$\boxed{10.4}$

$r = 1 - \cos\theta \qquad r' = \frac{d}{d\theta}(r) = 0 - (-\sin\theta) = \sin\theta$

$$\frac{dy}{dx} = \frac{\frac{d(y)}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}[r\sin\theta]}{\frac{d}{d\theta}[r\cos\theta]} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$$

$$\frac{dy}{d\theta} = r'\sin\theta + r\cos\theta = (\sin\theta)(\sin\theta) + (1-\cos\theta)\cos\theta$$

$$\frac{dy}{d\theta} = \sin^2\theta + (1-\cos\theta)\cos\theta = (1-\cos^2\theta) + (1-\cos\theta)\cos\theta$$

$$\frac{dy}{d\theta} = (1-\cos\theta)(1+\cos\theta) + (1-\cos\theta)\cos\theta$$

$$\frac{dy}{d\theta} = (1-\cos\theta)[1+\cos\theta+\cos\theta] = (1-\cos\theta)(1+2\cos\theta)$$

$$\boxed{\frac{dy}{d\theta} = (1-\cos\theta)(1+2\cos\theta)}$$

$\frac{dy}{d\theta} = 0 \quad \to \quad (1-\cos\theta) = 0 \text{ or } (1+2\cos\theta) = 0$

$1 = \cos\theta \qquad -\frac{1}{2} = \cos\theta$

$\theta = 0 \qquad \theta = \frac{2\pi}{3}, \ \theta = \frac{4\pi}{3}$

$\theta = 2\pi$

$$\frac{dx}{d\theta} = r'\cos\theta - r\sin\theta$$

$$\frac{dx}{d\theta} = (\sin\theta)\cos\theta - (1-\cos\theta)\sin\theta$$

$$\frac{dx}{d\theta} = \sin\theta[\cos\theta - (1-\cos\theta)]$$

$$\frac{dx}{d\theta} = \sin\theta[\cos\theta - 1 + \cos\theta] = \sin\theta(2\cos\theta - 1)$$

$\boxed{10.4}$

$$\frac{dx}{d\theta} = \sin\theta\,(2\cos\theta - 1)$$

$\frac{dx}{d\theta} = 0$, $\rightarrow \sin\theta = 0$, or $2\cos\theta - 1 = 0$

$\theta = 0$ | $2\cos\theta = 1$

$\theta = \pi$ | $\cos\theta = 1/2$

$\theta = \pi/3$

$\theta = 5\pi/3$

Horizontal Tangent:

$\frac{dy}{d\theta} = 0$ & $\frac{dx}{d\theta} \neq 0$

$\theta = 2\pi/3$ & $\theta = \frac{4\pi}{3}$

at $(\frac{3}{2}, 2\pi/3)$ & at $(\frac{3}{2}, 4\pi/3)$

Vertical tangent:

$\frac{dx}{d\theta} = 0$ & $\frac{dy}{d\theta} \neq 0$

$\theta = \pi$, $\theta = \pi/3$, & $\theta = 5\pi/3$

at $(2, \pi)$, at $(\frac{1}{2}, \pi/3)$ & at $(\frac{1}{2}, 5\pi/3)$

"Singular Points":

$\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} = 0$

$\theta = 0$ & $\theta = 2\pi$ $\leftarrow$ same point on cardioid, the "pole"

"cusp" $(0, 0)$

[10.4]

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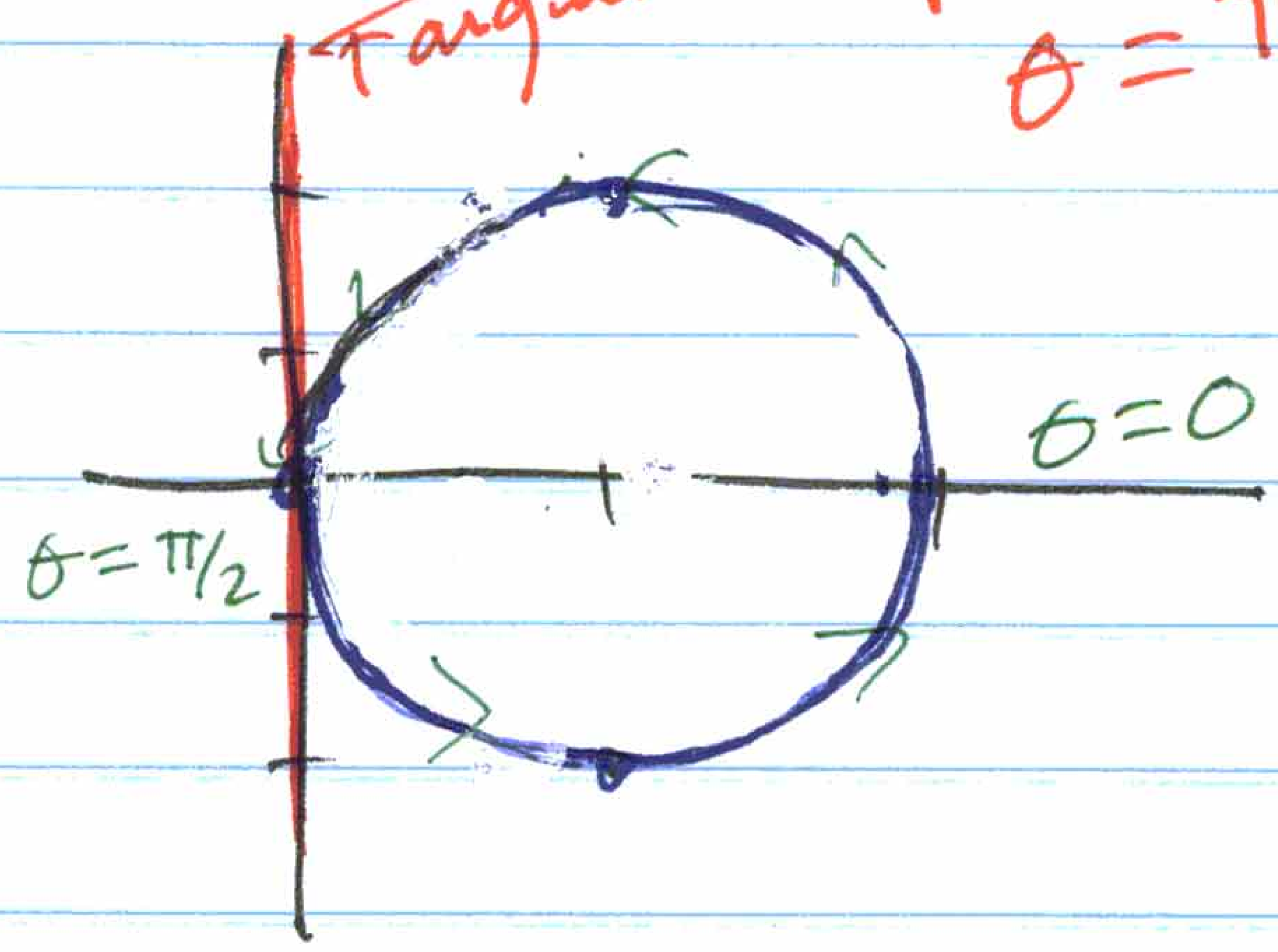
Find tangent(s) at pole for $r = 3\cos\theta$



Notice

$0 \le \theta \le \pi$

Tangent at pole: $r(\theta) = 0$ & $r'(\theta) \neq 0$.

$$r'(\theta) = \frac{d}{d\theta}(3\cos\theta)$$
$$= 3(-\sin\theta)$$
$$r'(\theta) = -3\sin\theta$$

$r(\theta) = 0$

$3\cos\theta = 0$

$\cos\theta = 0$

$\theta = \pi/2$

$$r'(\pi/2) = -3[\sin(\pi/2)]$$
$$= -3[1]$$
$$r'(\pi/2) = -3$$