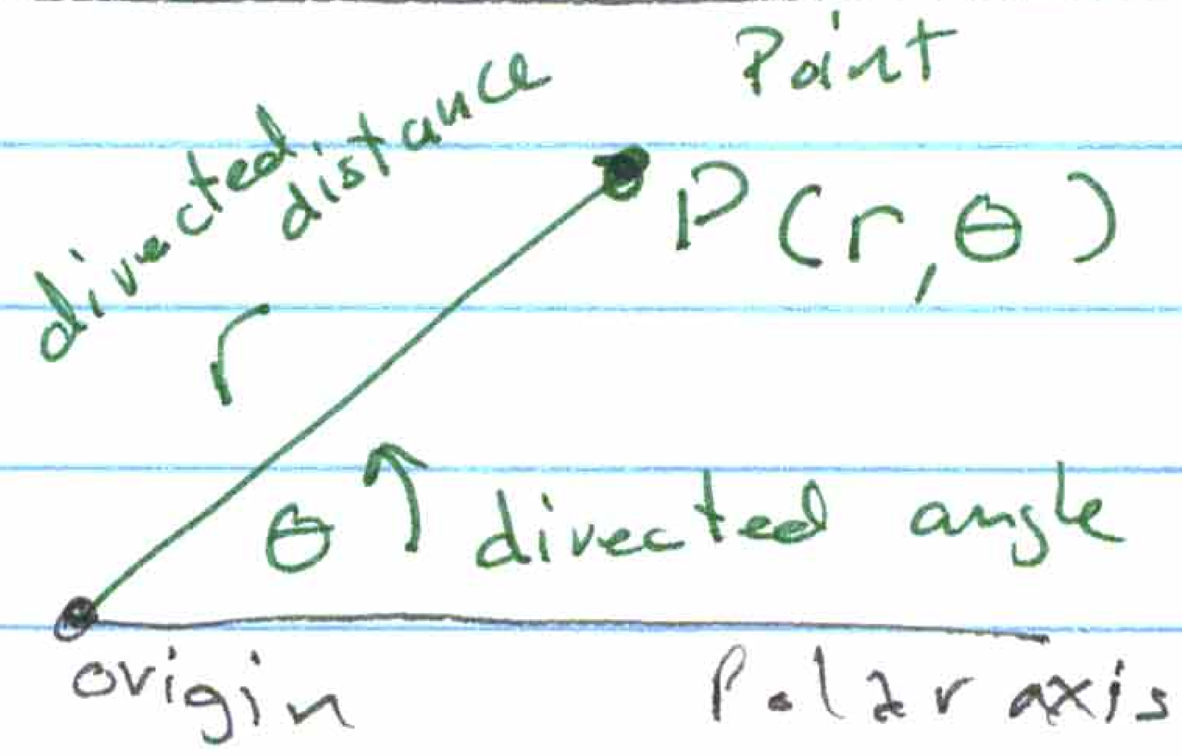
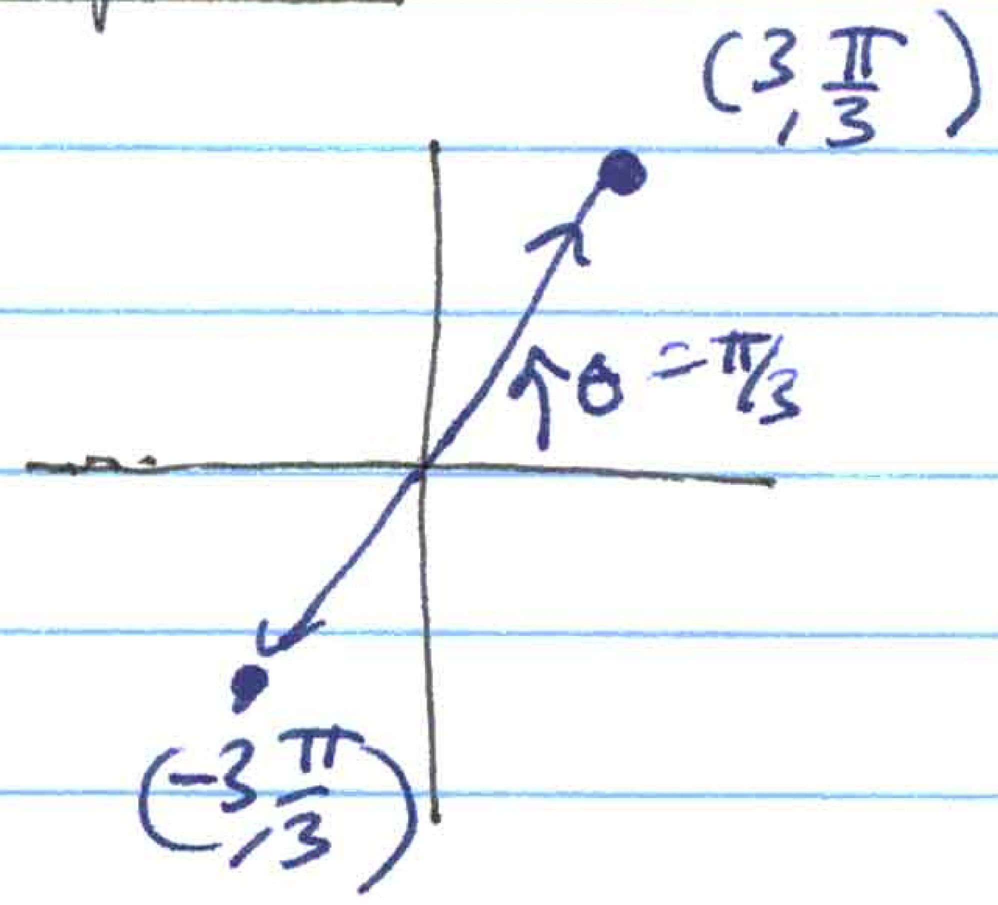
10.4 Polar Coordinates and Polar Graphs





Theorem 10.10 Coordinate Conversion

The polar coordinates (r, θ) of a point are related to the rectangular coordinate (x, y) of the point as follows

① $x = r\cos(\theta)$

$y = r\sin(\theta)$

② $\tan(\theta) = \dfrac{y}{x}$

$r^2 = x^2 + y^2$

Example: Convert $\left(\frac{5}{2}, \frac{4}{3}\right)$ rectangular into polar coordinates.

$r^2 = x^2 + y^2$

$r^2 = \left(\frac{5}{2}\right)^2 + \left(\frac{4}{3}\right)^2$

$r^2 = \frac{25}{4} + \frac{16}{9}$

$r^2 = \frac{289}{36}$

$r = +\sqrt{\frac{289}{36}}$

$\boxed{r = \frac{17}{6}}$

$\tan\theta = \dfrac{4/3}{5/2}$

$\tan\theta = \frac{4}{3} \cdot \frac{2}{5}$

$\tan\theta = \frac{8}{15}$

$\tan^{-1}(\tan\theta) = \tan^{-1}\left(\frac{8}{15}\right)$

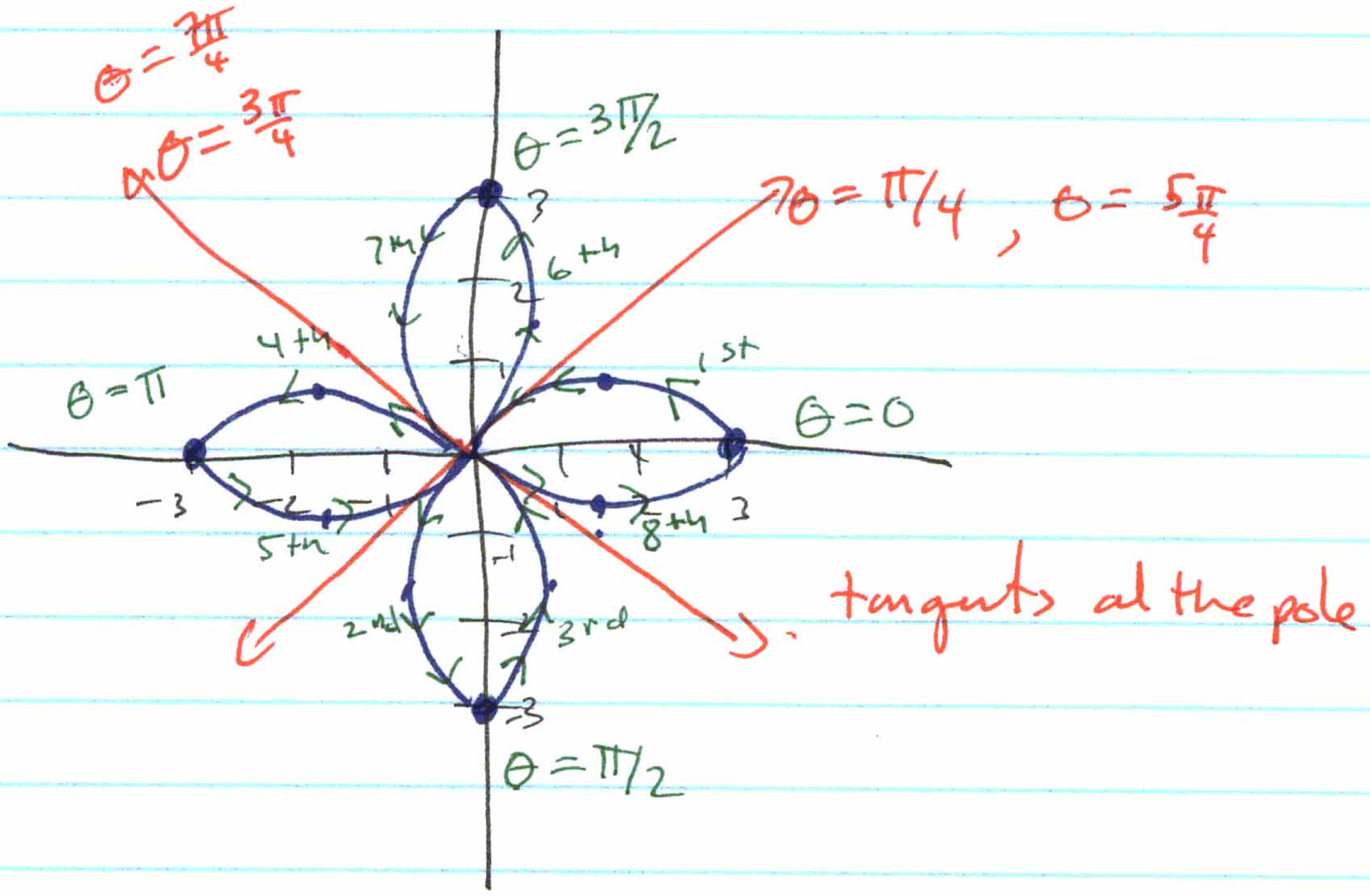
$\theta = \tan^{-1}\left(\frac{8}{15}\right)$

$\theta \approx 0.49$ Rad

$\left(\frac{5}{2}, \frac{4}{3}\right) \approx \left(\frac{17}{6}, 0.49 \text{ Rad}\right)$

10.4

#80 Sketch the graph of $r = 3\cos(2\theta)$ and find the tangents at the pole.



Solve: $r(\theta) = 0$

$$3\cos(2\theta) = 0$$

$$\cos(2\theta) = 0$$

$$\cos^{-1}[\cos(2\theta)] = \cos^{-1}(0)$$

$$2\theta = \pi/2$$

Give tangents at pole if $r'(\theta) \neq 0$.

$$\theta = \pi/4 \rightarrow \theta = \frac{3\pi}{4},$$

Same as other two

$\theta = \frac{5\pi}{4}$ $\theta = \frac{7\pi}{4}$

Find $r'(\theta) = \frac{d}{d\theta}[3\cos(2\theta)]$

$$r'(\theta) = 3[-\sin(2\theta)] \cdot \frac{d}{d\theta}(2\theta)$$

$$r'(\theta) = -3\sin(2\theta) \cdot 2$$

$$r'(\theta) = -6\sin(2\theta)$$

Check $r'(\pi/4) \neq 0$ →

$r'(\frac{\pi}{4}) = -6$
$r'(\frac{3\pi}{4}) = 6$
$r'(\frac{5\pi}{4}) = -6$
$r'(\frac{7\pi}{4}) = 6$

10.4

#34 Convert the equation from rectangular to polar coordinates and sketch its graph:

$$(x^2+y^2)^2 - 9(x^2-y^2)=0$$

$x = r\cos\theta$
$y = r\sin\theta$
$x^2+y^2=r^2$

$$(r^2)^2 - 9[r^2\cos^2\theta - r^2\sin^2\theta]=0$$
$$r^4 - 9r^2(\cos^2\theta - \sin^2\theta)=0$$
$$r^2[r^2 - 9\cos 2\theta]=0$$

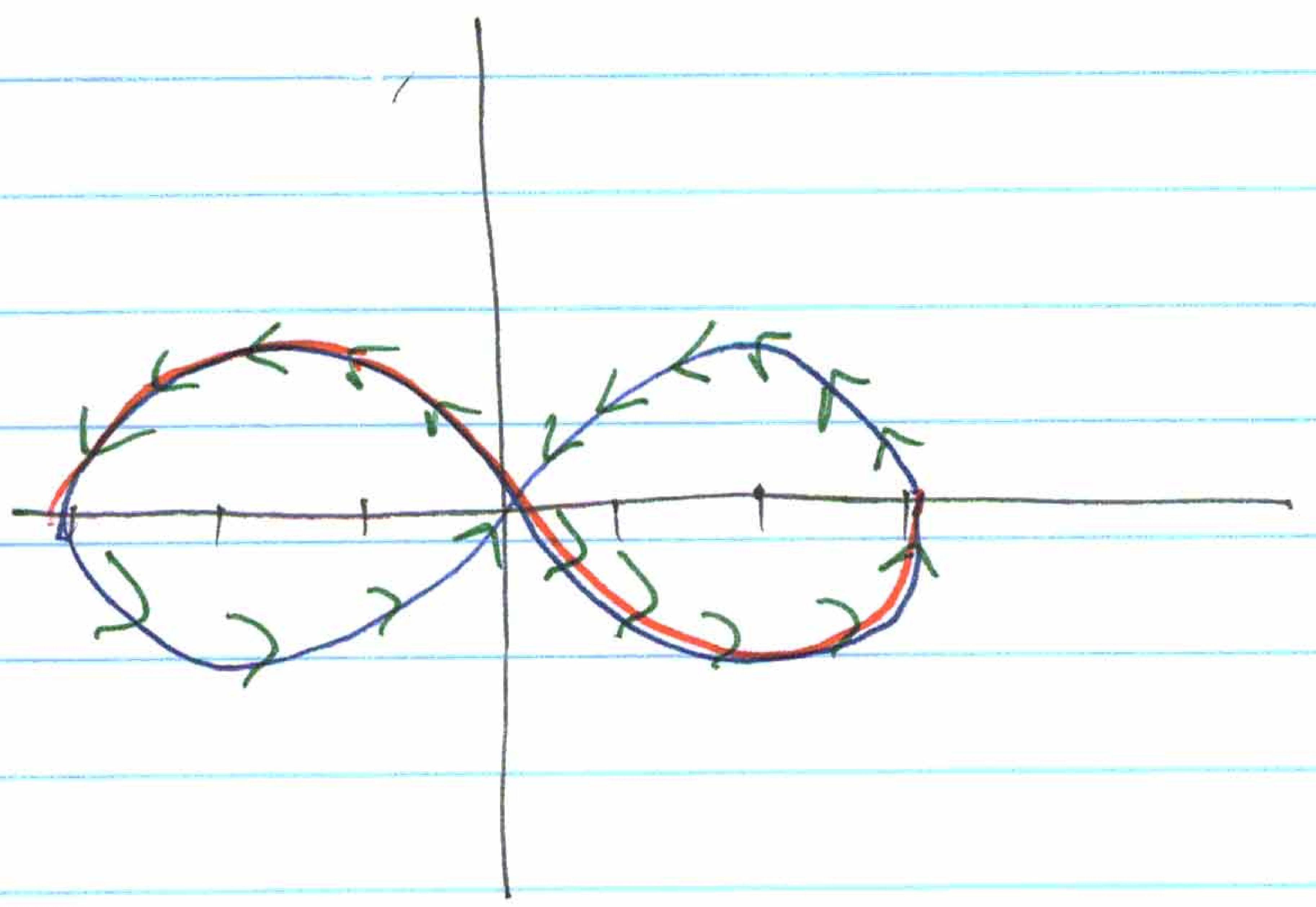
Either $r^2=0$, or $r^2 - 9\cos 2\theta = 0$

$r=0$ ← origin

$r^2 = 9\cos 2\theta$

If $\cos(2\theta) < 0$ there are no points!

If $\cos(2\theta) > 0$,
$r = \pm 3\sqrt{\cos(2\theta)}$



θ	r
0	3
$\frac{\pi}{12}$	$\frac{3\sqrt{3}}{2}$
$\pi/6$	$3/2$
$\pi/4$	0

$\frac{3\pi}{4} \le \theta \le \pi$

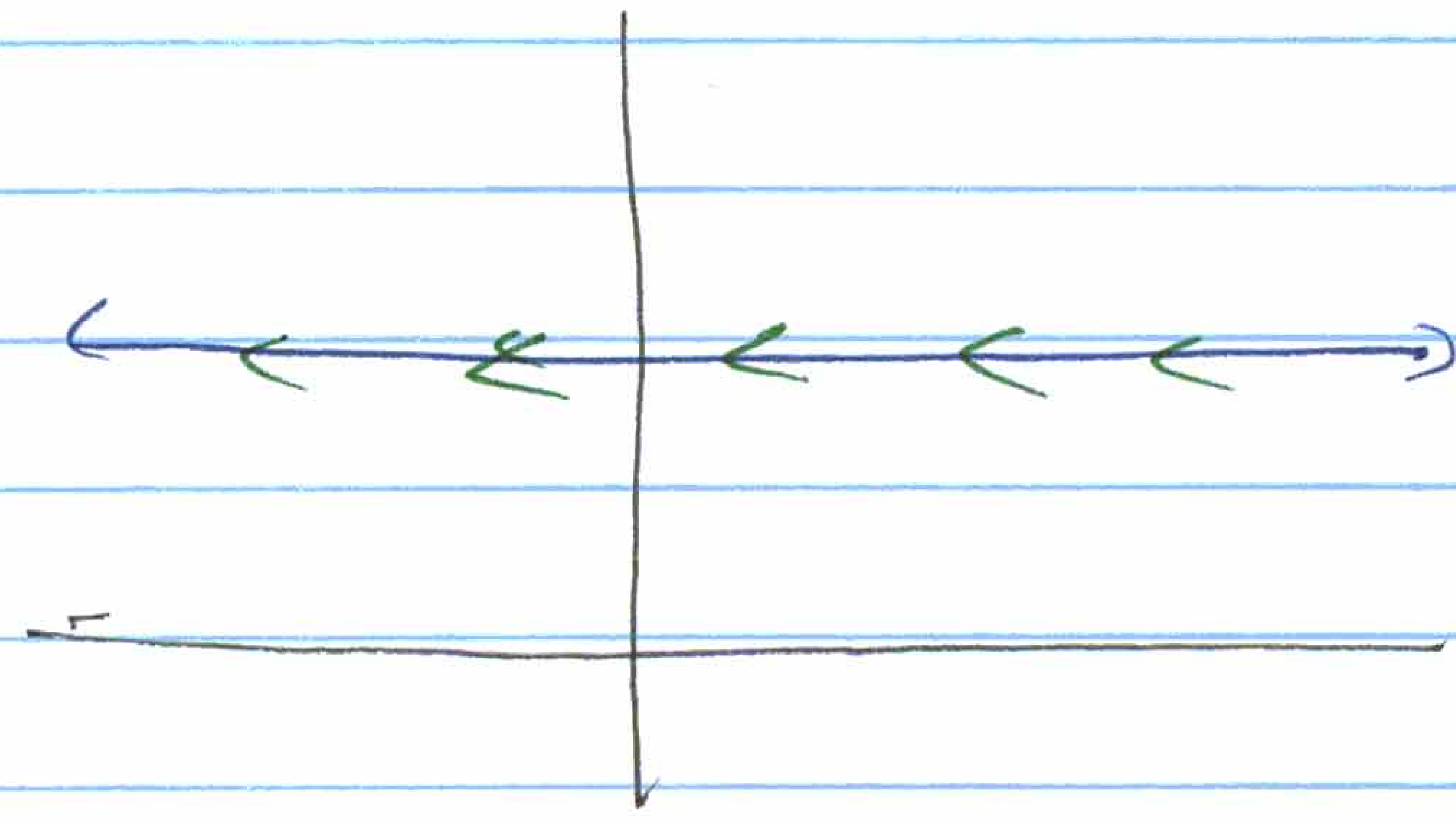
10.4

#42 Convert to polar coordinates and draw the graph

$$r = \frac{2}{\sin\theta}$$

$$r\sin\theta = 2$$

$$\boxed{y = 2}$$



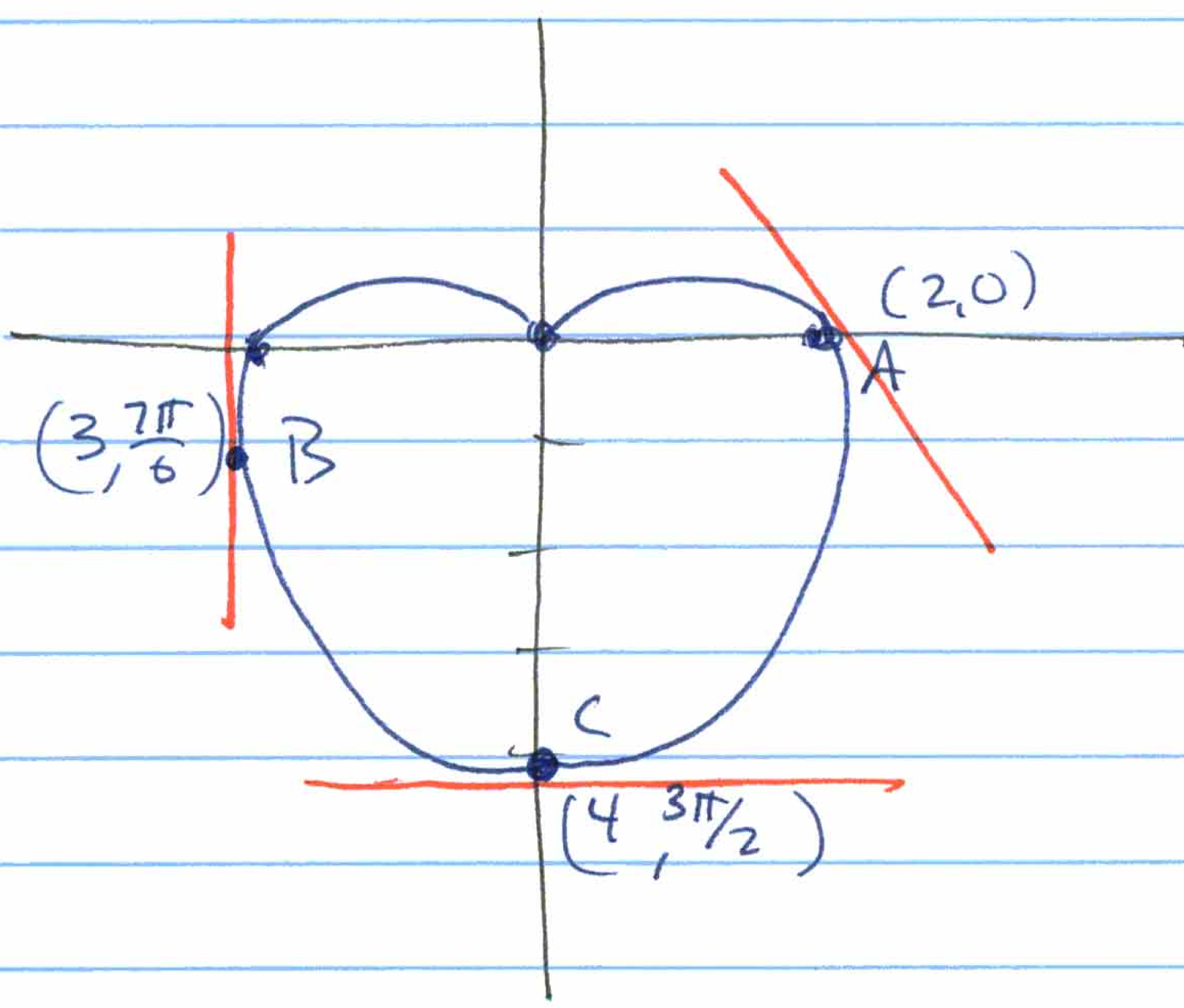
#60 Find $\frac{dy}{dx}$ and the slopes of the tangent lines at the given points:

A $(2, 0)$

B $(3, 7\pi/6)$

C $(4, 3\pi/2)$

$$r = 2(1 - \sin(\theta))$$



$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}[y]}{\frac{d}{d\theta}[x]}$$

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}[r\sin\theta]}{\frac{d}{d\theta}[r\cos\theta]}$$

$$\frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta + r(-\sin\theta)}$$

general Form

10.4

$r(\theta) = 2(1 - \sin\theta)$

$r(\theta) = 2 - 2\sin\theta$

$r'(\theta) = 0 - 2(\cos\theta)$

$r'(\theta) = -2\cos\theta$

$$\frac{dy}{dx} = \frac{r'(\theta)\cdot\sin\theta + r(\theta)\cdot\cos\theta}{r'(\theta)\cdot\cos\theta - r(\theta)\sin\theta}$$

$$\frac{dy}{dx} = \frac{(-2\cos\theta)(\sin\theta) + [2(1-\sin\theta)](\cos\theta)}{(-2\cos\theta)(\cos\theta) - [2(1-\sin\theta)](\sin\theta)}$$

$$\frac{dy}{dx} = \frac{-2\cos\theta\sin\theta + 2\cos\theta - 2\cos\theta\sin\theta}{-2\cos^2\theta - 2\sin\theta + 2\sin^2\theta}$$

$$\frac{dy}{dx} = \frac{2\cos\theta - 4\cos\theta\sin\theta}{-2[\cos^2\theta - \sin^2\theta + \sin\theta]} = \frac{2\cos\theta(1-2\sin\theta)}{-2[\cos^2\theta - \sin^2\theta + \sin\theta]}$$

$$\boxed{\frac{dy}{dx} = -\frac{\cos\theta(1-2\sin\theta)}{\cos^2\theta - \sin^2\theta + \sin\theta}}$$

at A $(2,0)$

$$\left.\frac{dy}{dx}\right|_{A(2,0)} = -\frac{\cos(0)[1-2\sin(0)]}{\cos^2(0) - \sin^2(0) + \sin(0)} = \frac{-(1)[1-0]}{1-0+0} = \frac{-1}{1}$$

$$\boxed{\left.\frac{dy}{dx}\right|_{A(2,0)} = -1}$$

at B $(3, 7\pi/6)$

$$\left.\frac{dy}{dx}\right|_{B(3,7\pi/6)} = -\frac{\cos(\frac{7\pi}{6})[1-2\sin(\frac{7\pi}{6})]}{\cos^2(\frac{7\pi}{6}) - \sin^2(\frac{7\pi}{6}) + \sin(\frac{7\pi}{6})} = -\frac{(-\frac{\sqrt{3}}{2})[1-2(-\frac{1}{2})]}{(-\frac{\sqrt{3}}{2})^2 - (-\frac{1}{2})^2 + (-\frac{1}{2})}$$

$$\boxed{\left.\frac{dy}{dx}\right|_{B(3,\frac{7\pi}{6})} = \frac{\frac{\sqrt{3}}{2}(2)}{\frac{3}{4} - \frac{1}{4} - \frac{2}{4}} = \frac{\sqrt{3}}{0} = \text{“}\infty\text{”}}$$

vertical tangent

at C $(4, \frac{3\pi}{2})$

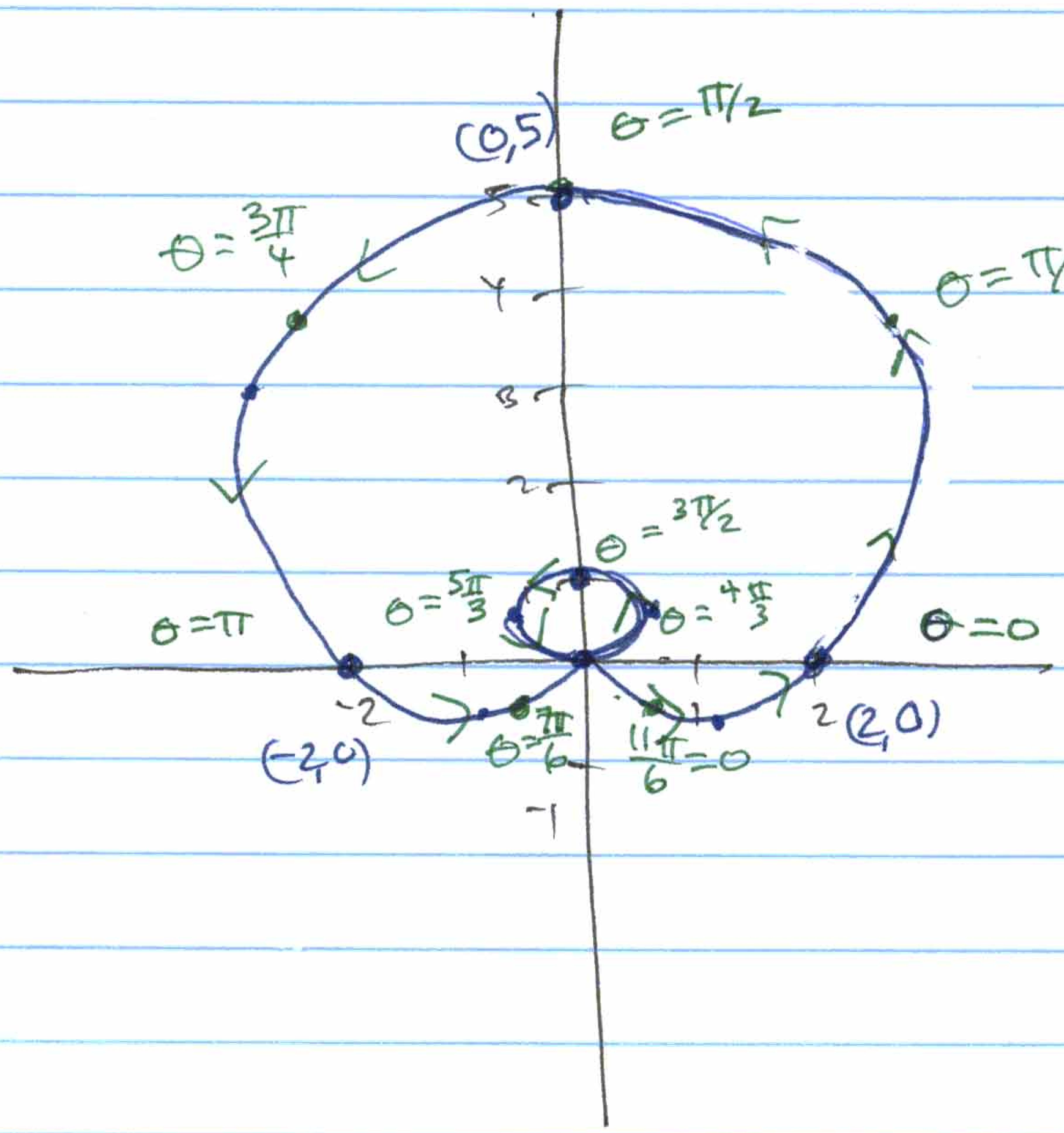
$$\left.\frac{dy}{dx}\right|_{C(4,\frac{3\pi}{2})} = -\frac{\cos(\frac{3\pi}{2})[1-2\sin(\frac{3\pi}{2})]}{\cos^2(\frac{3\pi}{2}) - \sin^2(\frac{3\pi}{2}) + \sin(\frac{3\pi}{2})} = \frac{0}{-2} = 0$$

($\cos(\frac{3\pi}{2}) \to 0$; $\sin^2(\frac{3\pi}{2}) \to (-1)^2$; $\sin(\frac{3\pi}{2}) \to -1$)

10.4

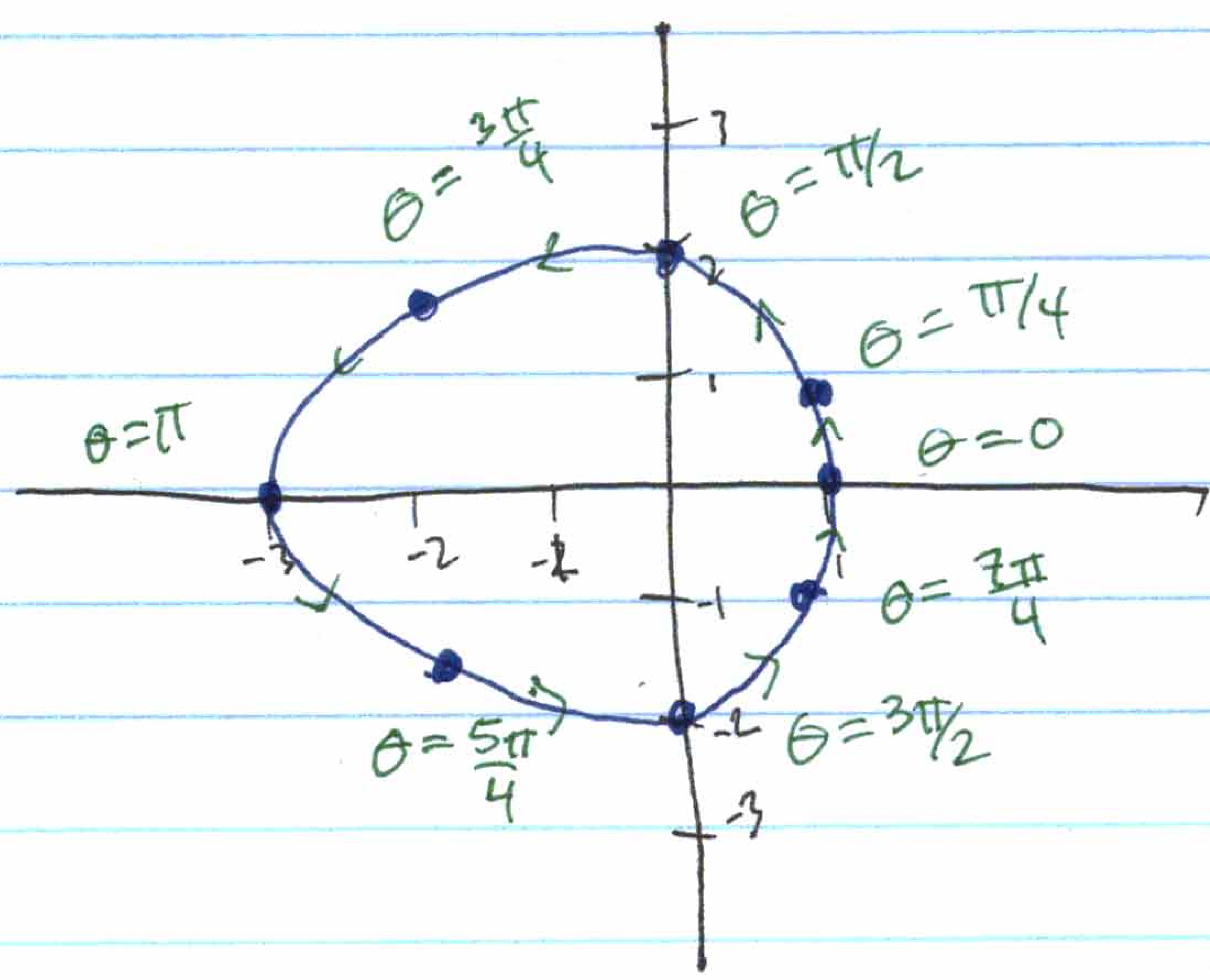
Example: Draw the graph

$r = 2 + 3\sin\theta$



θ	$\sin\theta$	$3\sin\theta$	$2+3\sin\theta$
0	0	0	2
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{3\sqrt{2}}{2}$	$\frac{3\sqrt{2}}{2}+2$
$\frac{\pi}{2}$	1	3	5
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{3\sqrt{2}}{2}$	$\frac{3\sqrt{2}}{2}+2$
π	0	0	2
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{3\sqrt{2}}{2}$	$-\frac{3\sqrt{2}}{2}+2$
$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{3\sqrt{3}}{2}$	$-\frac{3\sqrt{2}}{2}+2$
$\frac{3\pi}{2}$	-1	-3	-1
$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{3\sqrt{2}}{2}$	$-\frac{3\sqrt{2}}{2}+2$
$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{3\sqrt{2}}{2}$	$-\frac{3\sqrt{2}}{2}+2$
2π	0	0	2
$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$
$\frac{11\pi}{6}$	$-\frac{1}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$

Check page B5:



$r = 2 - \cos(\theta)$

$r = a \pm b\cos\theta$

$a = 2, \; b = 1$

$\frac{a}{b} \geq 2$

"Convex limaçon"

10.4 Theorem 10.11 – Slope in Polar Form

If f is a differiable function of θ, then the slope of the tangent line to the graph of $r = f(\theta)$ at the point (r, θ) is

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}$$

provided $\frac{dx}{d\theta} \neq 0$ at (r, θ).

Theorem 10.12 – Tangent Lines at the Pole

If $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, then the line $\theta = \alpha$ is tangent at the pole to the graph of $r = f(\theta)$.

10.4

#62 Use a graphing utility to

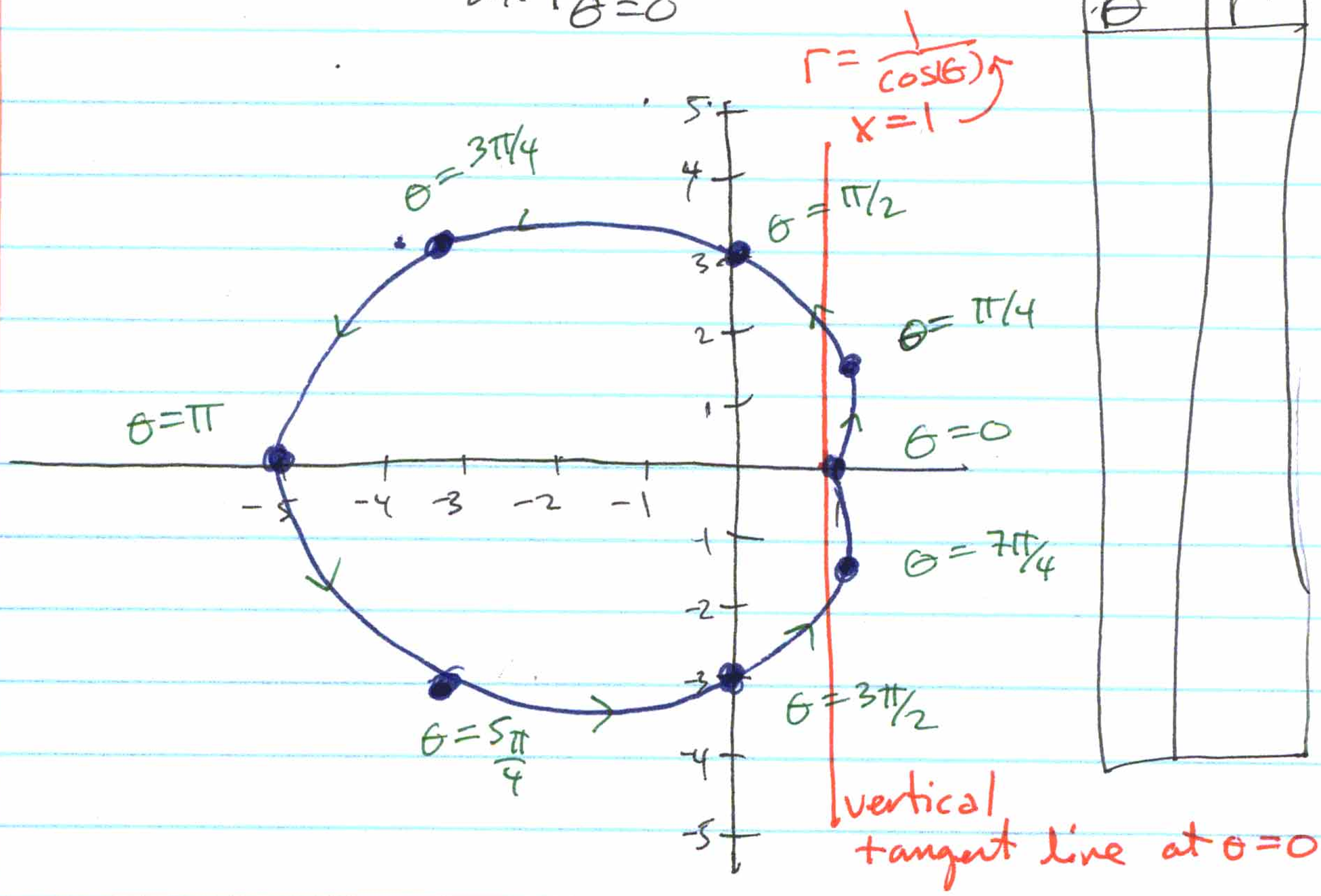
(a) graph the polar equation: $r(\theta) = 3 - 2\cos\theta$

(b) draw the tangent line at $\theta = 0$

(c) find $\left.\dfrac{dy}{dx}\right|_{\theta=0}$

Table??

θ	r



Looks like a vertical tangent line at $\theta = 0$.

Show that $\left.\dfrac{dx}{d\theta}\right|_{\theta=0} = 0$ and $\left.\dfrac{dy}{d\theta}\right|_{\theta=0} \neq 0$

$$\frac{dx}{d\theta} = \frac{d}{d\theta}[x]$$

$$= \frac{d}{d\theta}[r \cdot \cos\theta]$$

$$= r'\cos\theta - r\sin\theta$$

$$= (2\sin\theta)(\cos\theta) - (3 - 2\cos\theta)(\sin\theta)$$

$$= 2\sin\theta\cos\theta - [3\sin\theta - 2\sin\theta\cos\theta]$$

$$r'(\theta) = \frac{d}{d\theta}[3 - 2\cos\theta]$$

$$r'(\theta) = 0 - 2(-\sin\theta)$$

$$r'(\theta) = 2\sin\theta$$

[10.4]

Remember! $\left.\frac{dy}{dx}\right|_{\theta=0} = \left.\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right|_{\theta=0}$

#62 cont'd

$\frac{dx}{d\theta} = 2\sin\theta\cos\theta - 3\sin\theta + 2\sin\theta\cos\theta$

$\frac{dx}{d\theta} = 4\sin\theta\cos\theta - 3\sin\theta$

$\frac{dx}{d\theta} = \sin\theta\,(4\cos\theta - 3)$

$\left.\frac{dx}{d\theta}\right|_{\theta=0} = [\sin(0)][4\cos(0) - 3]$

$= 0\,[1]$

$\left.\frac{dx}{d\theta}\right|_{\theta=0} = 0$ ✓

Find $\frac{dy}{d\theta} = \frac{d}{d\theta}[y]$

$= \frac{d}{d\theta}[r\sin\theta]$

$= r'\sin\theta + r\cos\theta$

$= (2\sin\theta)(\sin\theta) + (3 - 2\cos\theta)(\cos\theta)$

$= 2\sin^2\theta + 3\cos\theta - 2\cos^2\theta$

$= 3\cos\theta - 2\cos^2\theta + 2\sin^2\theta$

$= 3\cos\theta - 2(\cos^2\theta - \sin^2\theta)$

$= 3\cos\theta - 2(2\cos^2\theta - 1)$

$\frac{dy}{d\theta} = 3\cos\theta - 4\cos^2\theta + 2$

$\left.\frac{dy}{d\theta}\right|_{\theta=0} = 3\cos(0) - 4\cos^2(0) + 2$

$= 3 - 4 + 2$

$\left.\frac{dy}{d\theta}\right|_{\theta=0} = 1 \neq 0$

this means that we have a point with a vertical tangent line at $\theta = 0$!

$\left.\frac{dy}{dx}\right|_{\theta=0}$ = undefined!