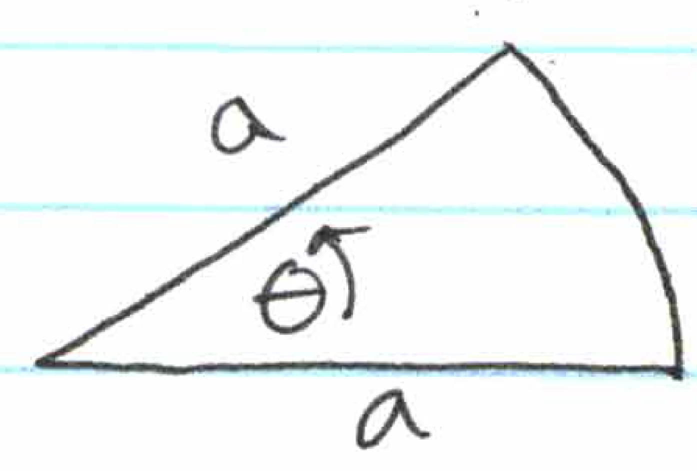
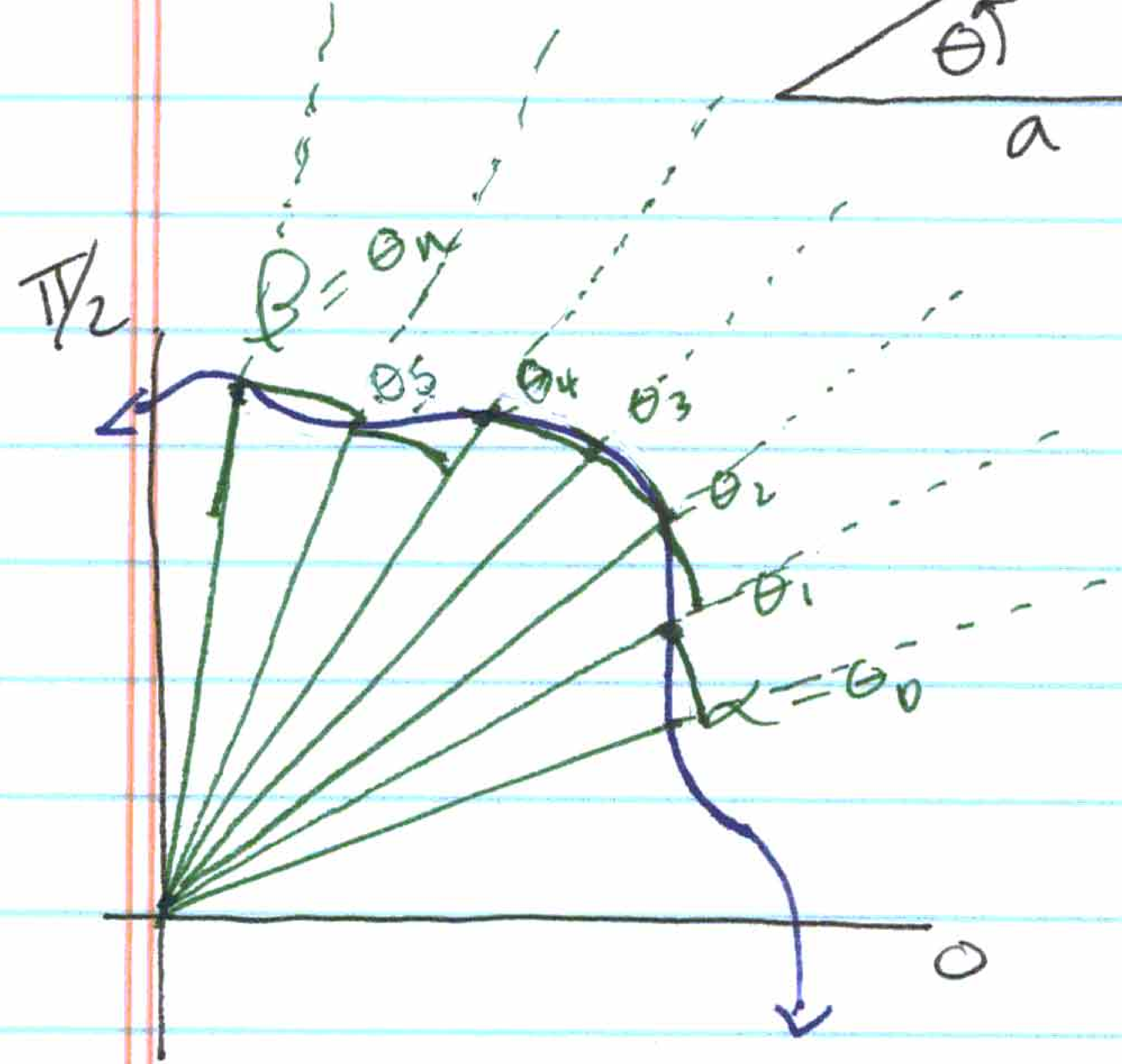
10.5 Area of a Polar Region & Arc Length in Polar Coordinates



Area of a Sector $= \pi a^2 \cdot \boxed{\frac{\theta}{2\pi}}$ Ratio of Sector to whole circle

$$A = \frac{a^2\theta}{2}$$



n sectors

$\frac{\beta-\alpha}{n} = \Delta\theta_i$ = central angle of ith sector

$r(\theta_i) = r_i$ is radius of ith sector

Area of ith sector

$$A_i = \frac{r_i^2 \Delta\theta_i}{2}$$

$$\text{Area} \approx \sum_{i=1}^{n} \frac{r_i^2 \Delta\theta_i}{2}$$

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{r_i^2 \Delta\theta_i}{2} = \int_\alpha^\beta \frac{r^2(\theta)\,d\theta}{2}$$

Theorem 10.13 - Area in Polar Coordinates

If f is continuous and nonnegative on the interval $[\alpha, \beta]$, $0 < \beta - \alpha \le 2\pi$, then the area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is given by

$$A = \frac{1}{2}\int_\alpha^\beta [f(\theta)]^2\,d\theta = \frac{1}{2}\int_\alpha^\beta r^2\,d\theta$$

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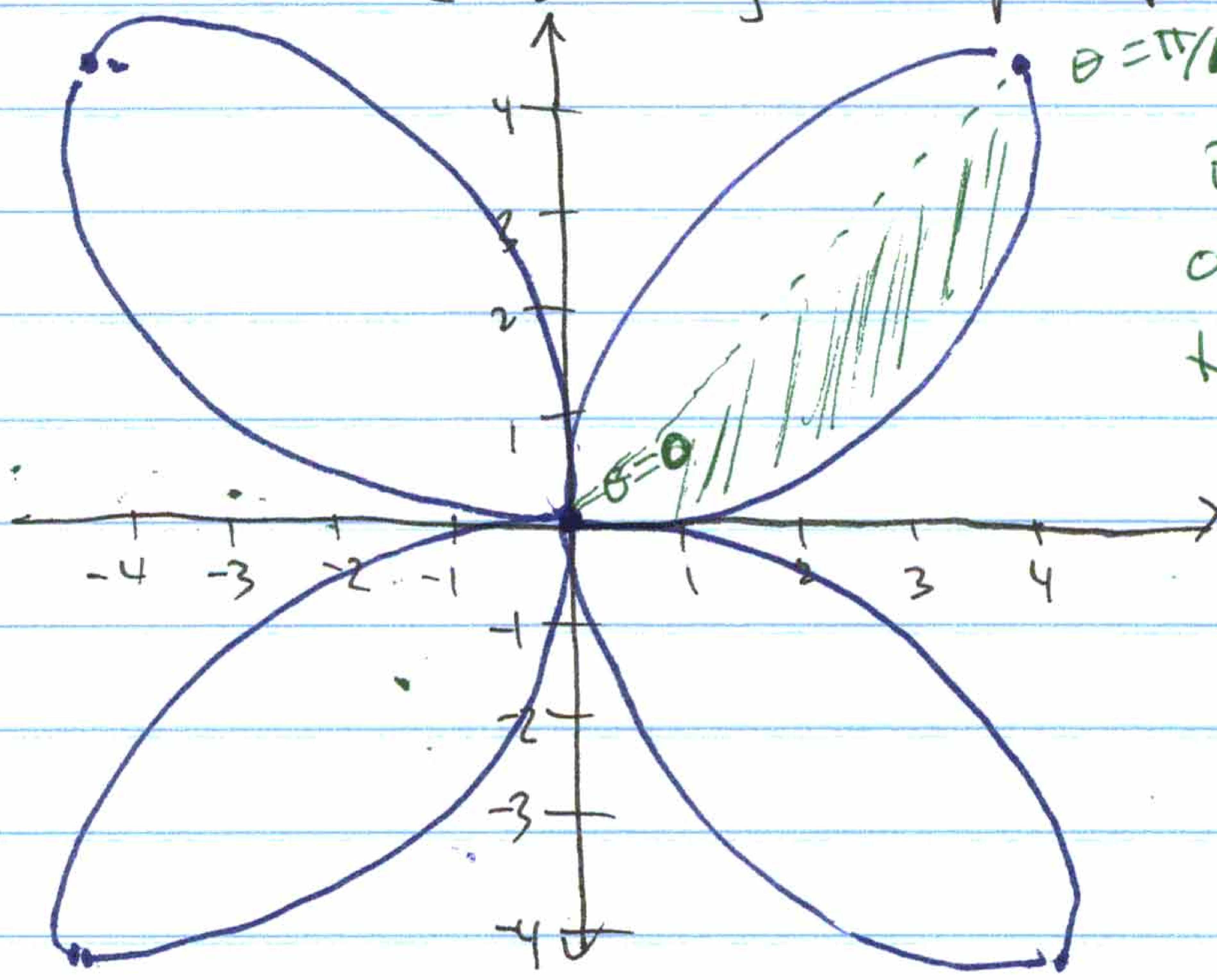
#8 Find the area of one petal of $r = 6\sin(2\theta)$



Find the area of $\frac{1}{2}$ one petal, then double this area. ← use symmetry of one petal.

$$\text{Area of one petal} = 2\cdot\left[\int_0^{\pi/4} \frac{[6\sin(2\theta)]^2}{2}\,d\theta\right]$$

$$= \int_0^{\pi/4} 36\sin^2(2\theta)\,d\theta$$

$$= 36\int_0^{\pi/4} \sin^2(2\theta)\,d\theta$$

use: $\sin^2 x = \frac{1-\cos(2x)}{2}$

$$= 36\int_0^{\pi/4} \left[\frac{1-\cos(4\theta)}{2}\right]d\theta$$

$$= 18\int_0^{\pi/4} [1-\cos(4\theta)]\,d\theta$$

$$= 18\left[\int_0^{\pi/4} 1\,d\theta - \int_0^{\pi/4} \cos(4\theta)\,d\theta\right]$$

$$= 18\left(\left[\theta - \frac{\sin(4\theta)}{4}\right]_0^{\pi/4}\right.$$

$\int \cos(4\theta)\,d\theta$

let $u = 4\theta$

$\frac{du}{d\theta} = 4$

$\frac{du}{4} = d\theta$

$\int \cos(u)\cdot\frac{du}{4}$

$= \frac{1}{4}\int \cos(u)\,du$

$= \frac{1}{4}(\sin u)$

$= \frac{\sin(4\theta)}{4}$

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~~#8 cont'd~~

$$\text{Area} = 18\left[\left(\frac{\pi}{4} - \frac{\sin[4(\frac{\pi}{4})]}{4}\right) - \left(0 - \frac{\sin(4\cdot 0)}{4}\right)\right]$$

$$= 18\left[\frac{\pi}{4} - 0 - 0\right]$$

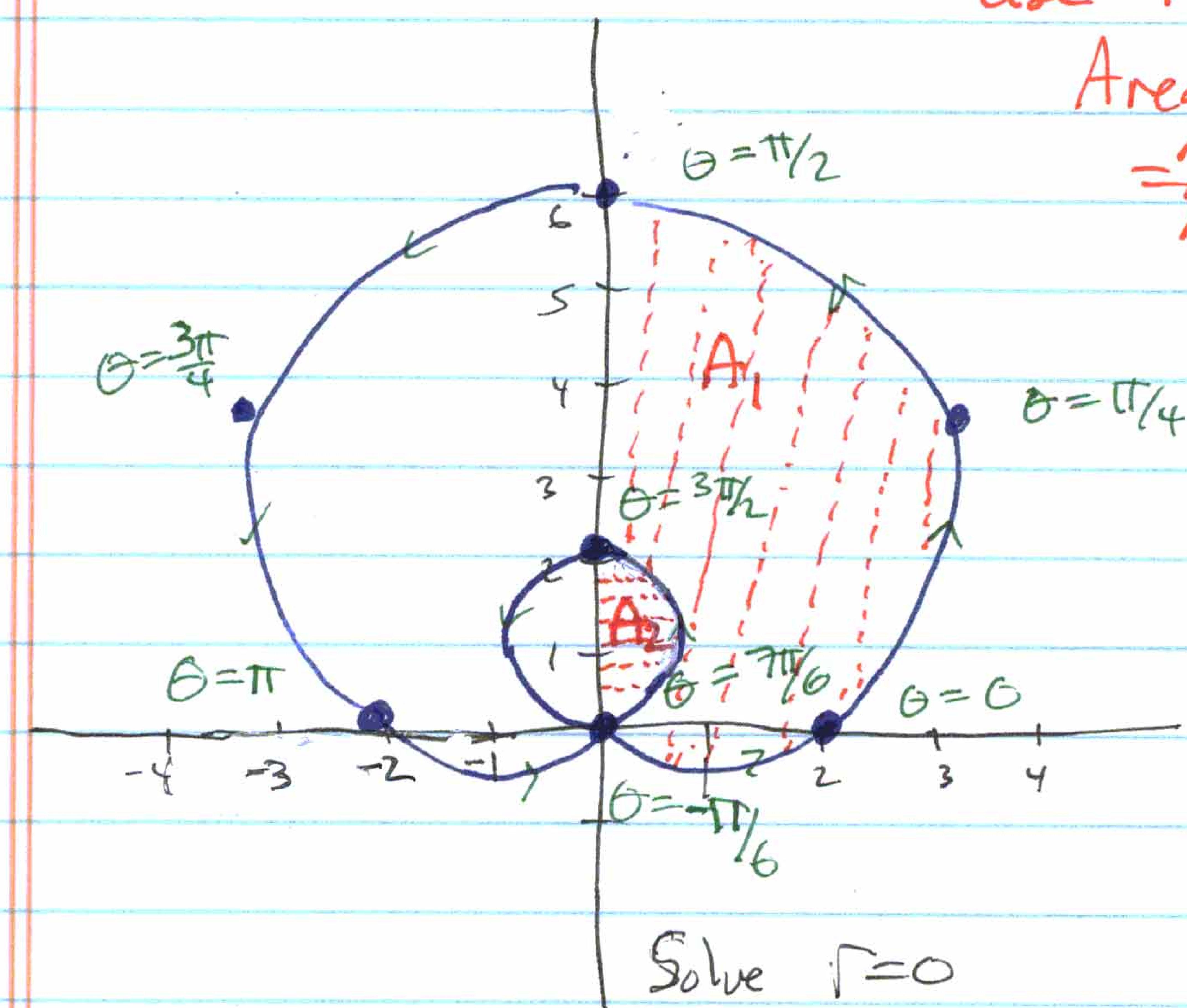
$$= 18\left(\frac{\pi}{4}\right)$$

$$= \frac{9}{2}\pi.$$

#16 Find the area between the loops of $r = 2(1+2\sin\theta)$

Use symmetry

Area Between $= 2[A_1 - A_2]$



Solve $r = 0$

$$2(1+2\sin\theta) = 0$$

$$1 + 2\sin\theta = 0$$

$$\sin\theta = \frac{-1}{2}$$

$\theta = \frac{7\pi}{6}$ $\quad$ $\theta = \frac{11\pi}{6}$ or $\theta = \frac{-\pi}{6}$

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Area Between loops $= 2 \cdot \left[\overbrace{\frac{1}{2}\int_{-\pi/6}^{\pi/2} [2(1+2\sin\theta)]^2 \, d\theta}^{A_1} - \overbrace{\frac{1}{2}\int_{7\pi/6}^{3\pi/2} [2(1+2\sin\theta)]^2 \, d\theta}^{A_2} \right]$

$= 2\left[(4\pi + 3\sqrt{3}) - (2\pi - 3\sqrt{3}) \right]$

$= 2 \cdot [2\pi + 6\sqrt{3}]$

$= 4\pi + 12\sqrt{3}$

$A_1 = \frac{1}{2}\int_{-\pi/6}^{\pi/2} [2(1+2\sin\theta)]^2 \, d\theta$

use $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$A_1 = \frac{1}{2}\int_{-\pi/6}^{\pi/2} 4[1 + 4\sin\theta + 4\sin^2\theta] \, d\theta$

$A_1 = 2\int_{-\pi/6}^{\pi/2} [1 + 4\sin\theta + 4\frac{(1-\cos 2\theta)}{2}] \, d\theta$

$A_1 = 2\int_{-\pi/6}^{\pi/2} [1 + 4\sin\theta + 2 - 2\cos(2\theta)] \, d\theta$

$A_1 = 2 \cdot \int_{-\pi/6}^{\pi/2} [3 + 4\sin\theta - 2\cos(2\theta)] \, d\theta$

$A_1 = 2 \cdot [3\theta + 4(-\cos\theta) - 2[\frac{1}{2}\sin(2\theta)]]_{-\pi/6}^{\pi/2}$

$A_1 = 2 \cdot [3\theta - 4\cos(\theta) - \sin(2\theta)]_{-\pi/6}^{\pi/2}$

$A_1 = 2[(3(\pi/2) - 4(0) - 0) -$
$(3(-\pi/6) - 4(\frac{\sqrt{3}}{2}) - (-\frac{\sqrt{3}}{2}))]$

$\int \cos(2\theta) \, d\theta$

$u = 2\theta$

$\frac{du}{d\theta} = 2$

$\frac{du}{2} = d\theta$

$= \int \cos(u) \cdot (\frac{du}{2})$

$= \frac{1}{2}\int \cos u \, du$

$= \frac{1}{2}\sin u$

$= \frac{1}{2}\sin(2\theta)$

$\boxed{10.5}$

$$A_1 = 2\left[\frac{3\pi}{2} + \frac{\pi}{2} + 2\sqrt{3} - \frac{\sqrt{3}}{2}\right]$$

$$A_1 = 3\pi + \pi + 4\sqrt{3} - \sqrt{3}$$

$$\boxed{A_1 = 4\pi + 3\sqrt{3}}$$

$$A_2 = \frac{1}{2}\int_{7\pi/6}^{3\pi/2} [2(1+2\sin\theta)]^2\, d\theta = \frac{1}{2}\int_{7\pi/6}^{3\pi/2} 4[1+4\sin\theta+4\sin^2\theta]\, d\theta$$

$$A_2 = 2\cdot\int_{7\pi/6}^{3\pi/2} [3 + 4\sin(\theta) - 2\cos(2\theta)]\, d\theta$$

$$A_2 = 2\cdot\left[3\theta + 4(-\cos\theta) - 2\left[\tfrac{1}{2}\sin 2\theta\right]\right]_{7\pi/6}^{3\pi/2}$$

$$A_2 = 2\cdot[3\theta - 4\cos\theta - \sin 2\theta]_{7\pi/6}^{3\pi/2}$$

$$A_2 = 2\left[\left(3\left(\tfrac{3\pi}{2}\right) - 4(0) - (0)\right) - \left(3\left(\tfrac{7\pi}{6}\right) - 4\left(-\tfrac{\sqrt{3}}{2}\right) - \left(\tfrac{\sqrt{3}}{2}\right)\right)\right]$$

$$A_2 = 2\left[\frac{9\pi}{2} - \frac{7\pi}{2} - 2\sqrt{3} + \frac{\sqrt{3}}{2}\right] = 2\left[\pi - 2\sqrt{3} + \frac{\sqrt{3}}{2}\right]$$

$$\boxed{A_2 = 2\pi - 3\sqrt{3}}$$

Yikes!

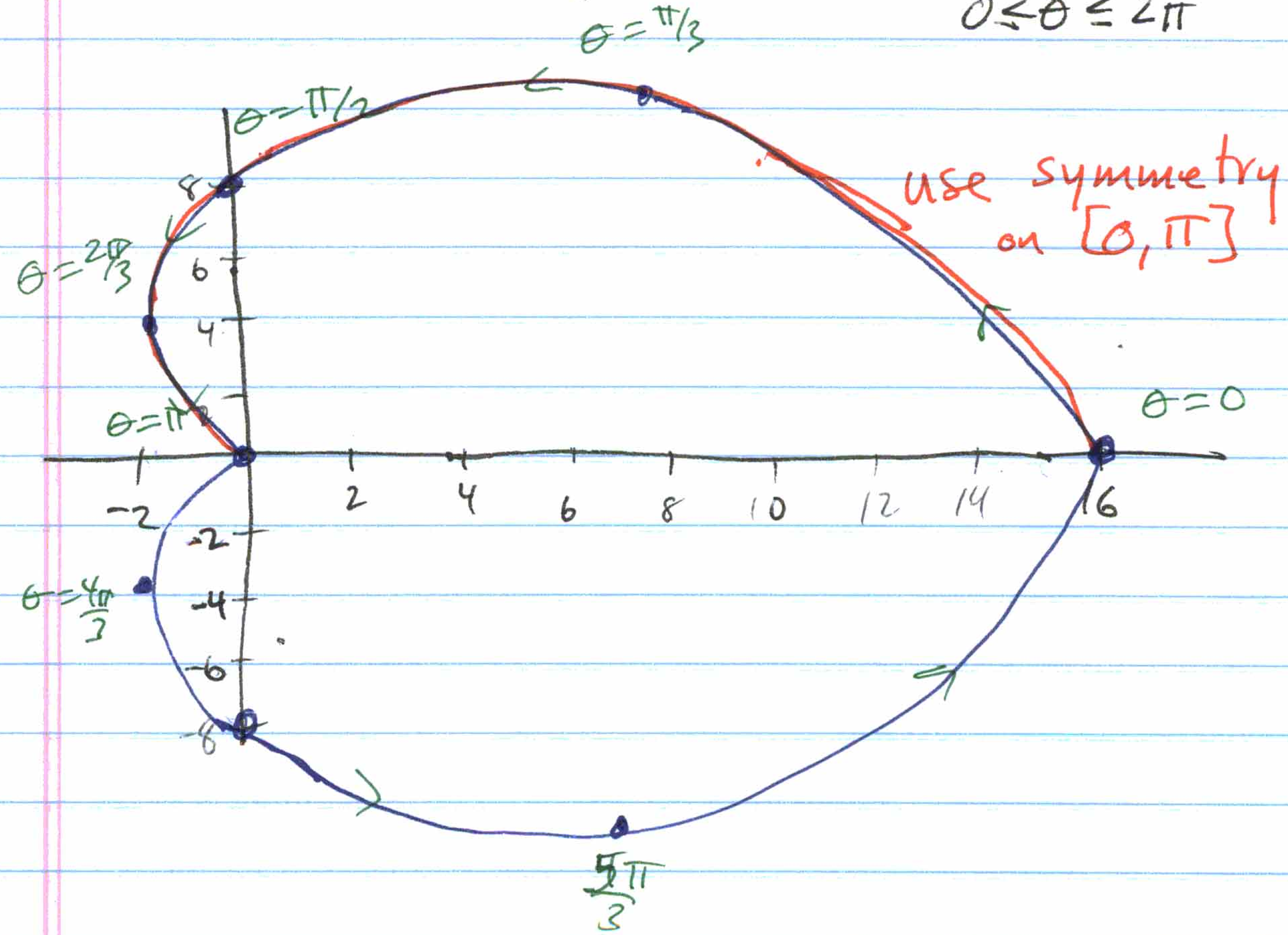
$\boxed{10.5}$

Theorem 10.14: Arc Length of a Polar Curve

Let f be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$.
The length of the graph $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$\text{Arc Length} = S = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta = \int_\alpha^\beta \sqrt{r^2 + \left[\frac{dr}{d\theta}\right]^2}\, d\theta$$

#48: Find the length of $r = 8(1 + \cos\theta)$ over $0 \le \theta \le 2\pi$



$$\text{Arc Length} = 2 \cdot \int_0^\pi \sqrt{r^2 + \left[\frac{dr}{d\theta}\right]^2}\, d\theta$$

symmetry

$$\frac{dr}{d\theta} = \frac{d}{d\theta}(r) = \frac{d}{d\theta}(8 + 8\cos\theta)$$

$$\frac{dr}{d\theta} = -8\sin\theta$$

$\boxed{10.5}$ #48 cont'd

$$\text{Arc Length} = 2\int_0^{\pi} \sqrt{[8+8\cos\theta]^2 + [-8\sin\theta]^2}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{64 + 128\cos\theta + 64\cos^2\theta + 64\sin^2\theta}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{64(1 + 2\cos\theta + \cos^2\theta + \sin^2\theta)}\, d\theta$$

$$= 16\int_0^{\pi} \sqrt{1 + 2\cos\theta + 1}\, d\theta$$

use

$$\cos^2 u = \frac{1+\cos 2u}{2}$$

$$2\cos^2 u = 1 + \cos 2u$$

$$2\cos^2\left(\tfrac{\theta}{2}\right) \;\; 1+\cos\theta$$

$$= 16\int_0^{\pi} \sqrt{2 + 2\cos\theta}\, d\theta$$

$$= 16\int_0^{\pi} \sqrt{2(1+\cos\theta)}\, d\theta$$

$$= 16\int_0^{\pi} \sqrt{2\left[2\cos^2\left(\tfrac{\theta}{2}\right)\right]}\, d\theta$$

$$= 16\int_0^{\pi} \sqrt{4\cos^2\left(\tfrac{\theta}{2}\right)}\, d\theta$$

$$= 16\int_0^{\pi} 2\cos\left(\tfrac{\theta}{2}\right) d\theta$$

let $z = \frac{\theta}{2}$

$\frac{dz}{d\theta} = \frac{1}{2}$

$2\,dz = d\theta$

$\theta = 0$	$\theta = \pi$
$z = \frac{0}{2}$	$z = \pi/2$
$z = 0$	

$$= 32\int_0^{\pi} \cos\left(\tfrac{\theta}{2}\right) d\theta$$

$$= 32\int_{z=0}^{z=\pi/2} \cos(z)\,(2\,dz)$$

$$= 64\int_0^{\pi/2} \cos(z)\,dz = 64\left[\sin z\right]_0^{\pi/2} = 64[(1) - 0]$$

$$= 64$$