8.2 Integration by Parts

Product Rule

$$\boxed{\frac{d}{dx}\left[f(x)\,g(x)\right] = f(x)\,g'(x) + g(x)\,f'(x)}$$

$$\frac{d}{dx}[uv] = u\cdot\frac{dv}{dx} + v\cdot\frac{du}{dx} = uv' + vu'$$

$$\int\left(\frac{d}{dx}[uv]\right)dx = \int\left(u\cdot\frac{dv}{dx}\right)dx + \int\left(v\cdot\frac{du}{dx}\right)dx$$

$$uv = \int u\,dv + \int v\,du$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Integration by PARTS

#28

$$\int \underset{u}{x}\,\sin(x)\,dx = \int u\,dv$$

$$= \left(\overset{u}{x}\right)\left(\overset{v}{-\cos(x)}\right) - \int\left(\overset{v}{-\cos(x)}\right)\left(\overset{du}{dx}\right)$$

$$= -x\cos(x) + \int \cos(x)\,dx$$

$$\boxed{= -x\cos(x) + \sin(x) + C}$$

let

$u = x$

$\frac{du}{dx} = 1$

$du = dx$

$\frac{dv}{dx} = \sin x$

$\int\left(\frac{dv}{dx}\right)dx = \int \sin x\,dx$

$\int dv = -\cos x$

$v = -\cos x$

check

$$\frac{d}{dx}\left(-x\cos(x) + \sin(x)\right) = -\frac{d}{dx}\left(x\cos(x)\right) + \frac{d}{dx}\left(\sin(x)\right)$$

$$= -\left[(x)\frac{d}{dx}(\cos(x)) + (\cos(x))\frac{d}{dx}(x)\right] + \cos(x)$$

$$= -\left[(x)(-\sin(x)) + \cos(x)(1)\right] + \cos(x) = x\sin(x) - \cos x + \cos x$$

$$= x\sin(x)$$ ✓

8.2

#8

$$\int \ln(3x)\,dx = \int 1\cdot \ln(3x)\,dx$$

let

$u = \ln(3x)$

$\frac{du}{dx} = \frac{1}{3x}\cdot\frac{d}{dx}(3x)$

$\frac{du}{dx} = \frac{1}{3x}\cdot 3$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{1}{x}dx$

$\frac{dv}{dx} = 1$

$dv = dx$

$\int dv = \int dx$

$v = x$

$$\int u\,dv = uv - \int v\,du$$

$$= \underbrace{(\ln(3x))}_{u}\underbrace{(\;x\;)}_{v} - \int \underbrace{(\;x\;)}_{v}\underbrace{(\tfrac{1}{x}dx)}_{du}$$

$$= x\ln(3x) - \int 1\,dx$$

$$\boxed{= x\ln(3x) - x + C}$$

#10

$$\int x^2\cos(x)\,dx$$

let

$u = x^2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\frac{dv}{dx} = \cos(x)$

$\int dv = \int \cos(x)\,dx$

$v = \sin(x)$

$$\int u\,dv = uv - \int v\,du$$

$$= (\;x^2\;)(\sin(x)) - \int(\sin(x))(2x\,dx)$$

$$= x^2\sin(x) - 2\underbrace{\int x\sin(x)\,dx}_{\#28,\text{ seen before??}}$$

let

$u = x$

$\frac{du}{dx} = 1$

$du = dx$

$\frac{dv}{dx} = \sin(x)$

$dv = \sin(x)\,dx$

$\int dv = \int \sin(x)\,dx$

$v = -\cos(x)$

$$= x^2\sin(x) - 2\left[(\;x\;)(-\cos(x)) - \int(-\cos(x))(dx)\right]$$

$$= x^2\sin(x) - 2\left[-x\cos(x) + \int \cos(x)\,dx\right]$$

$$= x^2\sin(x) - 2\left[-x\cos(x) + \sin x\right] + C$$

check??

$$\boxed{= x^2\sin(x) + 2x\cos(x) - 2\sin(x) + C}$$

$\boxed{8.2}$

#52

$$\int_0^1 x\cdot \arcsin(x^2)\,dx$$

$$= \left[\overset{u}{(\arcsin(x^2))}\overset{v}{\left(\frac{x^2}{2}\right)}\right]_{x=0}^{x=1} - \int_{x=0}^{x=1}\overset{v}{\left(\frac{x^2}{2}\right)}\overset{du}{\left(\frac{2x}{\sqrt{1-x^4}}\,dx\right)}$$

$$= \left[\frac{x^2}{2}\arcsin(x^2)\right]_0^1 - \int_0^1 \frac{x^3}{\sqrt{1-x^4}}\,dx$$

$$= \left(\frac{(1)^2}{2}\arcsin[(1)^2]\right) - (0) - \int_{w=1}^{w=0} \frac{x^3}{\sqrt{w}}\left(\frac{dw}{-4x^3}\right)$$

$$= \frac{1}{2}\arcsin(1) + \frac{1}{4}\int_{w=1}^{w=0} w^{-1/2}\,dw$$

$$= \frac{1}{2}\arcsin(1) + \frac{1}{4}\left[\frac{2}{1}w^{1/2}\right]_1^0$$

$$= \frac{1}{2}\arcsin(1) + \frac{1}{2}\left[(0)^{1/2} - (1)^{1/2}\right]$$

$$= \frac{1}{2}\arcsin(1) - \frac{1}{2}$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}$$

$$\boxed{= \frac{\pi}{4} - \frac{1}{2}}$$

$\boxed{\bigstar\ \arcsin(1) = \frac{\pi}{2}}$

let

$u = \arcsin(x^2)$

$\frac{du}{dx} = \frac{1}{\sqrt{1-(x^2)^2}}\cdot\frac{d}{dx}(x^2)$

$\frac{du}{dx} = \frac{1}{\sqrt{1-x^4}}\cdot 2x$

$\frac{du}{dx} = \frac{2x}{\sqrt{1-x^4}}$

$du = \frac{2x}{\sqrt{1-x^4}}\,dx$

$\frac{dv}{dx} = x$

$dv = x\,dx$

$\int dv = \int x\,dx$

$v = \frac{x^2}{2}$

---

let $w = 1 - x^4$

$\frac{dw}{dx} = -4x^3$

$\frac{dw}{-4x^3} = dx$

$x=0$, $w = 1-(0)^4$

$w = 1$

$x = 1$, $w = 1-(1)^4$

$w = 1-1$

$w = 0$

$\boxed{8.2}$

#6 $\int u\,dv$

$$\int x^2 e^{2x}\,dx$$

$$= (\overset{u}{x^2})(\overset{v}{\tfrac{1}{2}e^{2x}}) - \int (\overset{v}{\tfrac{1}{2}e^{2x}})(\overset{du}{2x\,dx})$$

$$= \frac{1}{2}x^2 e^{2x} - \underbrace{\int x e^{2x}\,dx}_{\int u\,dv = uv - \int v\,du}$$

$\downarrow$

$$= \frac{1}{2}x^2 e^{2x} - \left[(\overset{u}{x})(\overset{v}{\tfrac{1}{2}e^{2x}}) - \int (\overset{v}{\tfrac{1}{2}e^{2x}})(\overset{du}{dx}) \right]$$

$$= \frac{x^2}{2}e^{2x} - \left[\frac{x}{2}e^{2x} - \frac{1}{2}\int e^{2x}\,dx \right]$$

$$= \frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x} + \frac{1}{2}\int e^{2x}\,dx$$

$$= \frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x} + \frac{1}{2}\left[\frac{1}{2}e^{2x}\right] + C$$

$$\boxed{= \frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x} + \frac{1}{4}e^{2x} + C}$$

let

$u = x^2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\frac{dv}{dx} = e^{2x}$

$dv = e^{2x}\,dx$

$\int dv = \int e^{2x}\,dx$

$v = \int e^{z}\left(\frac{dz}{2}\right)$

let $z = 2x$

$\frac{dz}{dx} = 2$

$\frac{dz}{2} = dx$

$v = \frac{1}{2}\int e^{z}\,dz$

$v = \frac{1}{2}(e^{z})$

$v = \frac{1}{2}e^{2x}$

let

$u = x$

$\frac{du}{dx} = 1$

$du = dx$

let

$\frac{dv}{dx} = e^{2x}$

$dv = e^{2x}\,dx$

$\int dv = \int e^{2x}\,dx$

$v = \frac{1}{2}e^{2x}$

check??

$\boxed{8.2}$

#36 $\int u\,dv$

$$\int e^x \cos(2x)\,dx$$

$$= \overset{u}{(\cos(2x))}\overset{v}{(e^x)} - \int \overset{v}{(e^x)}\overset{du}{(-2\sin(2x)\,dx)}$$

$$= e^x\cos(2x) + 2\int e^x \sin(2x)\,dx$$

$$= e^x\cos(2x) + 2\left[\overset{u}{(\sin(2x))}\overset{v}{(e^x)} - \int \overset{v}{(e^x)}\overset{du}{(2\cos(2x)\,dx)}\right]$$

$$= e^x\cos(2x) + 2\left[e^x\sin(2x) - 2\int e^x\cos(2x)\,dx\right]$$

$$= e^x\cos(2x) + 2e^x\sin(2x) - 4\int e^x\cos(2x)\,dx$$

So, we have this again!!

$$\int e^x\cos(2x)\,dx = e^x\cos(2x) + 2e^x\sin(2x) - 4\int e^x\cos(2x)\,dx$$

$$+4\int e^x\cos(2x)\,dx \quad = \quad +4\int e^x\cos(2x)\,dx$$

$$5\int e^x\cos(2x)\,dx = e^x\cos(2x) + 2e^x\sin(2x)$$

$$\boxed{\int e^x\cos(2x)\,dx = \frac{1}{5}\left(e^x\cos(2x) + 2e^x\sin(2x)\right) + C}$$

let

$u = \cos(2x)$

$\frac{du}{dx} = -\sin(2x)\cdot\frac{d}{dx}(2x)$

$\frac{du}{dx} = -\sin(2x)\cdot 2$

$\frac{du}{dx} = -2\sin(2x)$

$du = -2\sin(2x)\,dx$

$\frac{dv}{dx} = e^x$

$dv = e^x\,dx$

$\int dv = \int e^x\,dx$

$v = e^x$

let

$u = \sin(2x)$

$\frac{du}{dx} = \cos(2x)\cdot\frac{d}{dx}(2x)$

$\frac{du}{dx} = 2\cos(2x)$

$du = 2\cos(2x)\,dx$

$\frac{dv}{dx} = e^x$

$dv = e^x\,dx$

$v = e^x$

Wow!!

"Loop"

$\boxed{8.2}$

#58

$$\int_0^{\pi/4} \overset{u}{x}\, \overset{dv}{\sec^2(x)dx}$$

$$= \Big[(\overset{u}{x})(\overset{v}{\tan(x)})\Big]_0^{\pi/4} - \int_0^{\pi/4} (\overset{v}{\tan(x)})(\overset{du}{dx})$$

$$= \Big[x\tan(x)\Big]_0^{\pi/4} - \int_0^{\pi/4} \tan(x)dx$$

$$= \left(\frac{\pi}{4}\right)\left(\tan\left(\frac{\pi}{4}\right)\right) - (0) - \Big[-\ln|\cos(x)|\Big]_0^{\pi/4}$$

$$= \frac{\pi}{4}\cdot 1 + \Big[\ln|\cos(x)|\Big]_0^{\pi/4}$$

$$= \frac{\pi}{4} + \left(\ln\left|\cos\left(\frac{\pi}{4}\right)\right| - \ln|\cos(0)|\right)$$

$$= \frac{\pi}{4} + \ln\left|\frac{\sqrt{2}}{2}\right| - \ln|1|$$

$$= \frac{\pi}{4} + \ln\left(\frac{\sqrt{2}}{2}\right) - 0$$

$$\boxed{= \frac{\pi}{4} + \ln\left(\frac{\sqrt{2}}{2}\right)} \quad \text{or}$$

$$= \frac{\pi}{4} + \ln\sqrt{2} - \ln 2$$

$$= \frac{\pi}{4} + \frac{1}{2}\ln 2 - \ln 2$$

$$\boxed{= \frac{\pi}{4} - \frac{1}{2}\ln 2}$$

let

$u = x$

$\frac{du}{dx} = 1$

$du = dx$

$\frac{dv}{dx} = \sec^2(x)$

$dv = \sec^2(x)dx$

$\int dv = \int \sec^2(x)dx$

$v = \tan(x)$

★ $\ln\sqrt{2} = \ln(2^{1/2})$

$= \frac{1}{2}\ln 2$

$\int u\,dv = uv - \int v\,du$

$\boxed{8.2}$ "Tabular Method"

#60

$\int x^3 e^{-2x}\,dx$

sign	Differentiate u	Integrate dv
$+$	x^3	e^{-2x}
$-$	$3x^2$	$-\frac{1}{2}e^{-2x}$
$+$	$6x$	$\frac{1}{4}e^{-2x}$
$-$	6	$-\frac{1}{8}e^{-2x}$
$+$	0	$\frac{1}{16}e^{-2x}$

$= +x^3\cdot\left(-\frac{1}{2}e^{-2x}\right) - 3x^2\cdot\left(\frac{1}{4}e^{-2x}\right) + 6x\cdot\left(-\frac{1}{8}e^{-2x}\right) - 6\cdot\left(\frac{1}{16}e^{-2x}\right) + C$

$$\boxed{= -\frac{x^3}{2}e^{-2x} - \frac{3}{4}x^2e^{-2x} - \frac{3}{4}xe^{-2x} - \frac{3}{8}e^{-2x} + C}$$

check??

$\int e^{-2x}\,dx$, let $z = -2x$, $\frac{dz}{dx} = -2$, $\frac{dz}{-2} = dx$

$= \int e^{z}\left(\frac{dz}{-2}\right)$

$= -\frac{1}{2}\int e^{z}\,dz$

$= -\frac{1}{2}e^{z}$

$= -\frac{1}{2}e^{-2x}$

$\int -\frac{1}{2}e^{-2x}\,dx$

$= -\frac{1}{2}\int e^{-2x}\,dx$

$= -\frac{1}{2}\left(-\frac{1}{2}e^{-2x}\right)$

$= \frac{1}{4}e^{-2x}$