$\boxed{8.3}$ Trigonometric Integrals

$m, n \geq 0$

Type I : $\int \sin^m(x) \cos^n(x)\, dx$

main tools : (a) $\sin^2 x + \cos^2 x = 1$

(b) $\sin^2 x = \dfrac{1 - \cos 2x}{2}$

(c) $\cos^2 x = \dfrac{1 + \cos 2x}{2}$

Strategies

(i) if m is odd, $m = 2k+1$

$$\int \sin^{2k+1} x \cos^n x\, dx = \int (\sin^2 x)^k \cos^n x \sin x\, dx$$

$$= \int (1 - \cos^2 x)^k \cos^n x \sin x\, dx$$

↑ expand & integrate

$$\left\{ \begin{array}{l} u = \cos x \\ \dfrac{du}{dx} = -\sin x \\ \dfrac{du}{-\sin x} = dx \end{array} \right.$$

(ii) if n is odd, $n = 2k+1$

$$\int \sin^m x \cos^{2k+1} x\, dx = \int \sin^m x (\cos^2 x)^k \cdot \cos x\, dx$$

$$= \int \sin^m x (1 - \sin^2 x)^k \cos x\, dx$$

↑ Expand & integrate

$$\left\{ \begin{array}{l} u = \sin x \\ \dfrac{du}{dx} = \cos x \\ \dfrac{du}{\cos x} = dx \end{array} \right.$$

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(iii) If m & n are both even, use identities (b) & (c) to reduce the powers.
Then, convert the integrand to odd powers of cosine. After this, proceed as with (ii).

#6

$$\int \cos^3(x)\sin^4(x)\,dx$$ ← n is odd (ii)

$\cos^2 x + \sin^2 x = 1$

$\cos^2 x = 1 - \sin^2 x$

Let $u = \sin x$

$\dfrac{du}{dx} = \cos x$

$\dfrac{du}{\cos x} = dx$

★ $\cos^2 x = 1 - u^2$

$$\int \cos^3(x)\sin^4(x)\,dx = \int \cos^2(x)\sin^4(x)\cos(x)\,dx$$

$$= \int (1-u^2)(u)^4\cos(x)\left(\frac{du}{\cos(x)}\right)$$

$$= \int (1-u^2)u^4\,du$$ ← Yes!

Now, expand & integrate!

$$= \int (u^4 - u^6)\,du$$

$$= \frac{u^5}{5} - \frac{u^7}{7} + C$$

$$\boxed{= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C}$$

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#12

$$\int \frac{\sin^5 t}{\sqrt{\cos t}}\,dt \quad \leftarrow \quad m \text{ is odd} \quad (i)$$

$\sin^2 x + \cos^2 x = 1$

$\sin^2 x = 1 - \cos^2 x$

let $u = \cos t$

$\frac{du}{dt} = -\sin t$

$\frac{du}{-\sin t} = dt$

★ $\sin^4 t = (\sin^2 t)^2$
$= (1-\cos^2 t)^2$
$= (1-u^2)^2$

$$\int \frac{\sin^5 t}{\sqrt{\cos t}}\,dt = \int \frac{\sin^4 t}{\sqrt{u}} \cdot \sin t\,dt$$

$$= \int (1-u^2)^2 u^{-1/2} \sin t \left(\frac{du}{-\sin t}\right)$$

$$= -\int (1-u^2)^2 u^{-1/2}\,du \quad \leftarrow \text{Yes!}$$

Now, expand & integrate!

$$= -\int (1-2u^2+u^4)u^{-1/2}\,du$$

$$= -\int \left(u^{-1/2} - 2u^{3/2} + u^{7/2}\right)du$$

$$= -\left(\frac{2}{1}u^{1/2} - 2\cdot\frac{2}{5}u^{5/2} + \frac{2}{9}u^{9/2}\right) + C$$

$$= -2u^{1/2} + \frac{4}{5}u^{5/2} - \frac{2}{9}u^{9/2} + C$$

$$\boxed{= -\frac{2}{9}\cos^{9/2}(t) + \frac{4}{5}\cos^{5/2}(t) - 2\cos^{1/2}(t) + C}$$

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#16 $\int \sin^4(2\theta)\,d\theta$ m is even, n=0 is even (iii)

use $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$

$$\sin^4(2\theta) = \left[\sin^2(2\theta)\right]^2$$
$$= \left[\frac{1-\cos(4\theta)}{2}\right]^2 = \left[\frac{1-\cos(4\theta)}{2}\right]\left[\frac{1-\cos(4\theta)}{2}\right]$$
$$= \frac{1}{4}\left[1-2\cos(4\theta)+\cos^2(4\theta)\right]$$

So, $$\int \sin^4(2\theta)\,d\theta = \frac{1}{4}\int \left[1-2\cos(4\theta)+\cos^2(4\theta)\right]d\theta$$

use
$\cos^2(x) = \dfrac{1+\cos(2x)}{2}$
$\cos^2(4\theta) = \dfrac{1+\cos(8\theta)}{2}$

$$= \frac{1}{4}\int\left[1-2\cos(4\theta) + \underbrace{\frac{1+\cos(8\theta)}{2}}\right]d\theta$$
$$= \frac{1}{4}\int\left[\frac{2}{2}-2\cos(4\theta) + \frac{1}{2} + \frac{\cos(8\theta)}{2}\right]d\theta$$
$$= \frac{1}{4}\int\left[\frac{3}{2}-2\cos(4\theta) + \frac{\cos(8\theta)}{2}\right]d\theta$$
$$= \frac{1}{4}\left[\int \frac{3}{2}\,d\theta - 2\underbrace{\int\cos(4\theta)\,d\theta} + \frac{1}{2}\underbrace{\int\cos(8\theta)\,d\theta}\right]$$

let $u = 4\theta$
$\frac{du}{d\theta} = 4$
$\frac{du}{4} = d\theta$

let $z = 8\theta$
$\frac{dz}{d\theta} = 8$
$\frac{dz}{8} = d\theta$

$$= \frac{1}{4}\left[\frac{3\theta}{2} - 2\int\cos(u)\left(\frac{du}{4}\right) + \frac{1}{2}\int\cos(z)\left(\frac{dz}{8}\right)\right]$$
$$= \frac{1}{4}\left[\frac{3\theta}{2} - \frac{1}{2}\int\cos(u)\,du + \frac{1}{16}\int\cos(z)\,dz\right]$$

Yes!

$$= \frac{1}{4}\left[\frac{3\theta}{2} - \frac{1}{2}\left[\sin(u)\right] + \frac{1}{16}\left[\sin(z)\right]\right] + C$$

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#16 cont'd

$$= \frac{1}{4}\left[\frac{3}{2}\theta - \frac{1}{2}\sin(4\theta) + \frac{1}{16}\sin(8\theta)\right] + C$$

$$\boxed{= \frac{3}{8}\theta - \frac{1}{8}\sin(4\theta) + \frac{1}{64}\sin(8\theta) + C}$$

check??

Type II: $\int \tan^n(x)\sec^m(x)\,dx$

main tools:

(a) $1 + \tan^2(x) = \sec^2(x)$

(b) $\tan^2(x) = \sec^2(x) - 1$

(c) $\frac{d}{dx}[\tan(x)] = \sec^2(x)$

(d) $\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$

$\boxed{\text{strategies}}$

(i) if m is even, $m = 2k$

$$\int \sec^m(x)\tan^n(x)\,dx = \int \sec^{2k}(x)\tan^n(x)\,dx$$

$$= \int \sec^{2k-2}(x)\tan^n(x)\cdot\sec^2(x)\,dx$$

$$= \int [\sec^2(x)]^{k-1}\tan^n(x)\sec^2(x)\,dx$$

$$= \int \underbrace{[1+\tan^2(x)]^{k-1}\cdot\tan^n(x)}_{\text{Expand \& integrate}}\sec^2(x)\,dx$$

let $u = \tan(x)$

$\frac{du}{dx} = \sec^2(x)$

$\frac{du}{\sec^2(x)} = dx$

(ii) if n is odd, $n = 2k+1$

$$\int \tan^n(x)\sec^m(x)\,dx = \int \tan^{2k+1}(x)\sec^m(x)\,dx$$

$$= \int \tan^{2k}(x)\sec^{m-1}(x)\cdot\sec(x)\tan(x)\,dx$$

$$= \int [\tan^2(x)]^k \sec^{m-1}(x)\cdot\sec(x)\tan(x)\,dx$$

$$= \int \underbrace{[\sec^2(x)-1]^k \sec^{m-1}(x)}_{\text{Expand \& integrate}}\cdot\sec(x)\tan(x)\,dx$$

let $u = \sec(x)$

$\frac{du}{dx} = \sec(x)\tan(x)$

$\frac{du}{\sec(x)\tan(x)} = dx$

(iii) if n is even & $m=0$,

$$\int \tan^n(x)\,dx = \int \tan^{n-2}(x)\tan^2(x)\,dx$$

$$= \int \tan^{n-2}(x)[\sec^2(x)-1]\,dx$$

$$= \int \tan^{n-2}(x)\sec^2(x)\,dx - \int \tan^{n-2}(x)\,dx$$

use (i)

repeat if needed to reduce tangent powers using $\tan^2(x) = \sec^2(x) - 1$

(iv) if m is odd & $n=0$, use "Parts"

$m=2k+1$

$$\int \sec^m(x)\,dx = \int \sec^{2k+1}(x)\,dx$$

$$= \int \sec(x)\cdot\sec^{2k}(x)\,dx = \int u\,dv$$

$$= uv - \int v\,du$$

let $u=\sec(x)$

$$\frac{du}{dx} = \sec(x)\tan(x)$$

$$du = \sec(x)\tan(x)\,dx$$

$$dv = \sec^{2k}(x)\,dx$$

$$\int dv = \int \sec^{2k}(x)\,dx$$

$$v = \int \sec^{2k-2}(x)\cdot\sec^2(x)\,dx$$

$$v = \int [\sec^2(x)]^{k-1}\sec^2(x)\,dx$$

$$v = \int [1+\tan^2(x)]^{k-1}\sec^2(x)\,dx$$

↑ use (i)

i.e. let $z=\tan(x)$

$$\frac{dz}{dx} = \sec^2(x)$$

$$\frac{dz}{\sec^2(x)} = dx$$

$$v = \int [1+z^2]^{k-1}\sec^2(x)\left(\frac{dz}{\sec^2(x)}\right)$$

$$v = \int \left((1+z^2)^{k-1}\right)dz$$

↑ expand & integrate

(v) if (i) → (iv) do not apply → try converting to sines & cosines then look for simplification

Example

m is even → (i)

$$\int \sec^4\left(\frac{x}{2}\right)\tan\left(\frac{x}{2}\right)dx$$

let $u = \tan\left(\frac{x}{2}\right)$

$$\frac{du}{dx} = \sec^2\left(\frac{x}{2}\right)\cdot\frac{d}{dx}\left(\frac{x}{2}\right)$$

$$\frac{du}{dx} = \sec^2\left(\frac{x}{2}\right)\cdot\frac{1}{2}$$

$$\frac{2\,du}{\sec^2\left(\frac{x}{2}\right)} = dx$$

$$= \int \sec^2\left(\frac{x}{2}\right)\tan\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right)dx$$

$$= \int \left[1+\tan^2\left(\frac{x}{2}\right)\right]\tan\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right)dx$$

use $\sec^2(\theta) = 1+\tan^2(\theta)$

$$= \int [1+u^2](u)\sec^2\left(\frac{x}{2}\right)\left(\frac{2\,du}{\sec^2\left(\frac{x}{2}\right)}\right)$$

$$= 2\int (1+u^2)u\,du$$

← YES! , Expand & integrate

$$= 2\int (u+u^3)\,du$$

$$= 2\left[\frac{u^2}{2}+\frac{u^4}{4}\right]+C$$

$$= u^2+\frac{u^4}{2}+C$$

$$\boxed{= \tan^2\left(\frac{x}{2}\right)+\frac{\tan^4\left(\frac{x}{2}\right)}{2}+C}$$

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#32

n is odd $\rightarrow$ (ii)

$$\int \tan^3\left(\frac{\pi x}{2}\right)\sec^2\left(\frac{\pi x}{2}\right)dx$$

let $u = \sec\left(\frac{\pi}{2}x\right)$

$$\frac{du}{dx} = \sec\left(\frac{\pi}{2}x\right)\tan\left(\frac{\pi x}{2}\right)\cdot\frac{d}{dx}\left(\frac{\pi x}{2}\right)$$

$$\frac{du}{dx} = \sec\left(\frac{\pi x}{2}\right)\tan\left(\frac{\pi x}{2}\right)\cdot\frac{\pi}{2}$$

$$\frac{2}{\pi}\left(\frac{du}{\sec\left(\frac{\pi x}{2}\right)\tan\left(\frac{\pi x}{2}\right)}\right) = dx$$

use $\tan^2(\theta) = \sec^2(\theta) - 1$

$$= \int \tan^2\left(\frac{\pi x}{2}\right)\sec\left(\frac{\pi x}{2}\right)\cdot\sec\left(\frac{\pi x}{2}\right)\tan\left(\frac{\pi x}{2}\right)dx$$

$$= \int\left[\sec^2\left(\frac{\pi x}{2}\right) - 1\right]\sec\left(\frac{\pi x}{2}\right)\cdot\sec\left(\frac{\pi x}{2}\right)\tan\left(\frac{\pi x}{2}\right)dx$$

$$= \int\left[u^2 - 1\right](u)\cdot\sec\left(\frac{\pi x}{2}\right)\tan\left(\frac{\pi x}{2}\right)\left(\frac{2}{\pi}\cdot\frac{du}{\sec\left(\frac{\pi x}{2}\right)\tan\left(\frac{\pi x}{2}\right)}\right)$$

$$= \frac{2}{\pi}\int\left(u^3 - u\right)du \quad \leftarrow \text{Yes!!} \quad \text{Expand \& Integrate!}$$

$$= \frac{2}{\pi}\left[\frac{u^4}{4} - \frac{u^2}{2}\right] + C$$

$$= \frac{1}{\pi}\left[\frac{u^4}{2} - u^2\right] + C$$

$$\boxed{= \frac{1}{\pi}\left[\frac{1}{2}\sec^4\left(\frac{\pi}{2}x\right) - \sec^2\left(\frac{\pi}{2}x\right)\right] + C}$$

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Example

n is even, $m=0$ $\rightarrow$ (iii)

$$\int \tan^6(x)\,dx$$

use $\tan^2(x) = \sec^2(x) - 1$

Repeat if needed

Reduce the powers

let $u = \tan(x)$

$\frac{du}{dx} = \sec^2(x)$

$\frac{du}{\sec^2(x)} = dx$

$$= \int \tan^4(x)\tan^2(x)\,dx$$

$$= \int \tan^4(x)\left[\sec^2(x) - 1\right]dx$$

$$= \int \tan^4(x)\sec^2(x)\,dx - \int \tan^4(x)\,dx$$

$$= \int u^4 \sec^2(x)\left(\frac{du}{\sec^2(x)}\right) - \int \tan^2(x)\tan^2(x)\,dx$$

$$= \int u^4\,du - \int \tan^2(x)\left[\sec^2(x) - 1\right]dx$$

$$= \frac{u^5}{5} - \int \tan^2(x)\sec^2(x)\,dx + \int \tan^2(x)\,dx$$

$$= \frac{\tan^5(x)}{5} - \int u^2\sec^2(x)\left(\frac{du}{\sec^2(x)}\right) + \int (\sec^2(x) - 1)\,dx$$

$$= \frac{\tan^5(x)}{5} - \int u^2\,du + \int \sec^2(x)\,dx - \int 1\,dx$$

$$= \frac{\tan^5(x)}{5} - \frac{u^3}{3} + \tan(x) - x + c$$

$$\boxed{= \frac{\tan^5(x)}{5} - \frac{\tan^3(x)}{3} + \tan(x) - x + c}$$

Wow!!

"Parts" $\rightarrow$ $\int u\,dv = uv - \int v\,du$

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Example: m is odd, $n=0$, $\rightarrow$ (iv), & use "Parts"

$$\int \sec^3(x)\,dx$$

$$= \int \sec(x)\sec^2(x)\,dx = \int u\,dv$$

$$= (\overset{u}{\ \ })(\overset{v}{\ \ }) - \int (\overset{v}{\ \ })(\overset{du}{\ \ })$$

let $u = \sec(x)$

$\frac{du}{dx} = \sec(x)\tan(x)$

$du = \sec(x)\tan(x))$

let $dv = \sec^2(x)\,dx$

$\int dv = \int \sec^2(x)\,dx$

$v = \tan(x)$

$$= (\sec(x))(\tan(x)) - \int (\tan(x))(\sec(x)\tan(x)\,dx)$$

$$= \sec(x)\tan(x) - \int \sec(x)\tan^2(x)\,dx$$

, use $\tan^2(x) = \sec^2(x) - 1$

$$= \sec(x)\tan(x) - \int \sec(x)[\sec^2(x) - 1]\,dx$$

$\boxed{\int \sec(x)\,dx = \ln|\sec(x)+\tan(x)| + C}$

$$= \sec(x)\tan(x) - \int \sec^3(x)\,dx + \int \sec(x)\,dx$$

$$= \sec(x)\tan(x) + \ln|\sec(x)+\tan(x)| - \int \sec^3(x)\,dx$$

"Loop" ★

So, Add

$$\int \sec^3(x)\,dx = \sec(x)\tan(x) + \ln|\sec(x)+\tan(x)| - \int \sec^3(x)\,dx$$

$$+\int \sec^3(x)\,dx \quad = \quad +\int \sec^3(x)\,dx$$

$$\frac{1}{2}\cdot 2\int \sec^3(x)\,dx = [\sec(x)\tan(x) + \ln|\sec(x)+\tan(x)|]\cdot\frac{1}{2}$$

$$\boxed{\int \sec^3(x)\,dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x)+\tan(x)| + C}$$

Zoinks!!

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#42

(V) convert to sines & cosines

$$\int \frac{\tan^2(x)}{\sec^5(x)}\,dx = \int \frac{\left(\frac{\sin(x)}{\cos(x)}\right)^2}{\left(\frac{1}{\cos(x)}\right)^5}\,dx$$

$$= \int \frac{\sin^2(x)}{\cos^2(x)} \cdot \frac{\cos^5(x)}{1}\,dx$$

odd, split the cosines

$$= \int \sin^2(x)\cos^3(x)\,dx$$

$$= \int \sin^2(x)\cdot[\cos^2(x)]\cos(x)\,dx$$

$$= \int \sin^2(x)\,[1-\sin^2(x)]\cos(x)\,dx$$

let $u = \sin(x)$

$\frac{du}{dx} = \cos(x)$

$\frac{du}{\cos(x)} = dx$

use $\cos^2(x) = 1-\sin^2(x)$

$$= \int (u)^2[1-(u)^2]\cos(x)\left(\frac{du}{\cos(x)}\right)$$

$$= \int (u^2 - u^4)\,du$$

$$= \frac{u^3}{3} - \frac{u^5}{5} + C$$

$$\boxed{= \frac{1}{3}\sin^3(x) - \frac{1}{5}\sin^5(x) + C}$$

Type III: Sine - Cosine Products with Different Angles

(a) $\sin(\alpha)\sin(\beta) = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$

(b) $\cos(\alpha)\cos(\beta) = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$

(c) $\sin(\alpha)\cos(\beta) = \frac{1}{2}[\sin(\alpha-\beta) + \sin(\alpha+\beta)]$

Product-to-Sum Formulas

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#54

$$\int \sin(-4x)\cos(3x)\,dx$$

use

$\sin(\alpha)\cos(\beta) = \frac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)]$

$$= \frac{1}{2}\int [\sin(-7x) + \sin(-x)]\,dx$$

$$= \frac{1}{2}\int \sin(-7x)\,dx + \frac{1}{2}\int \sin(-x)\,dx$$

$$= -\frac{1}{2}\int \sin(7x)\,dx - \frac{1}{2}\int \sin(x)\,dx \quad \star\star$$

$$= -\frac{1}{2}\int \sin(u)\left(\frac{du}{7}\right) - \frac{1}{2}\left[-\cos(x)\right] + C$$

$$= \frac{-1}{14}\int \sin(u)\,du + \frac{1}{2}\cos(x) + C$$

$$= \frac{-1}{14}\left[-\cos(u)\right] + \frac{1}{2}\cos(x) + C$$

$$= \frac{1}{14}\cos(7x) + \frac{1}{2}\cos(x) + C$$

$\alpha = -4x$, $\beta = 3x$

$\alpha - \beta = -4x - 3x$

$= -7x$

$\alpha + \beta = -4x + 3x$

$= -x$

sine is odd $\star\star$

$\sin(-x) = -\sin(x)$

cosine is even $\star\star$

$\cos(-x) = \cos(x)$

let $u = 7x$

$\frac{du}{dx} = 7$

$\frac{du}{7} = dx$