★ ★ Keys: $\sqrt{a^2-u^2}$ ★ ★

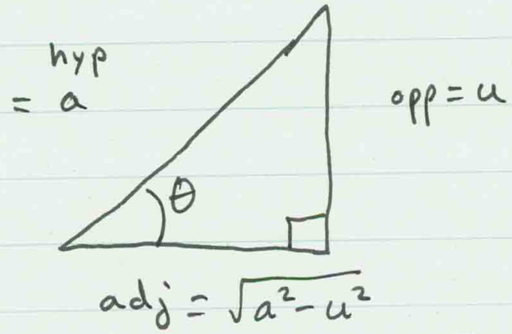
$\sqrt{a^2+u^2}$

$\sqrt{u^2-a^2}$

$\boxed{a>0}$

$\boxed{8.4}$ Trigonometric Substitutions

Type I: $\sqrt{a^2-u^2}$



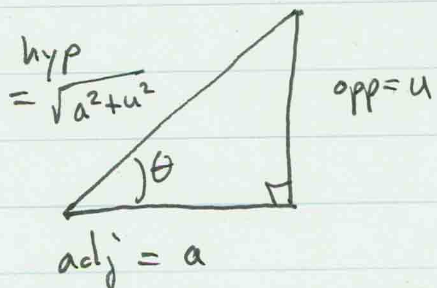
Let $u = a\sin(\theta)$

$\frac{u}{a} = \sin(\theta)$

$\frac{u}{a} = \frac{opp}{hyp}$, $\frac{\sqrt{a^2-u^2}}{a} = \frac{adj}{hyp}$

$-\frac{\pi}{2} \le \theta \le \pi/2$ & $\sqrt{a^2-u^2} = a\cos(\theta)$

Type II: $\sqrt{a^2+u^2}$



Let $u = a\tan(\theta)$

$\frac{u}{a} = \tan(\theta)$

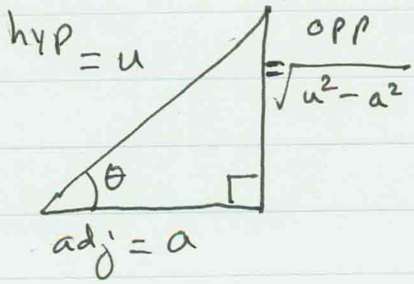
$\frac{u}{a} = \frac{opp}{adj}$

$\frac{\sqrt{a^2+u^2}}{a} = \frac{hyp}{adj}$

$-\pi/2 < \theta < \pi/2$

& $\sqrt{a^2+u^2} = a\sec(\theta)$

Type III: $\sqrt{u^2-a^2}$



let $u = a\sec(\theta)$

$\frac{u}{a} = \sec(\theta)$

$\frac{u}{a} = \frac{hyp}{adj}$, $\frac{\sqrt{u^2-a^2}}{a} = \frac{opp}{adj}$

$\sqrt{u^2-a^2} = a\tan(\theta)$

$0 \le \theta < \pi/2$, or

$\pi/2 < \theta \le \pi$

If $u > a$, use $\sqrt{u^2-a^2} = a\tan(\theta)$

If $u < -a$, use $\sqrt{u^2-a^2} = -a\tan(\theta)$

$\boxed{8.4}$

#22

$$\int \frac{x}{\sqrt{9-x^2}}\,dx$$

$$= \int \frac{[3\sin(\theta)]}{\sqrt{9-(3\sin(\theta))^2}}\left(3\cos(\theta)\,d\theta\right)$$

$$= 9\int \frac{\sin(\theta)\cos(\theta)\,d\theta}{\sqrt{9-9\sin^2\theta}}$$

$\cos^2(\alpha) + \sin^2(\alpha) = 1$

$\cos^2(\alpha) = 1 - \sin^2(\alpha)$

$9\cos^2(\alpha) = 9 - 9\sin^2(\alpha)$

use $\sqrt{}$

$3\cos(\alpha) = \sqrt{9-9\sin^2(\alpha)}$

$$= 9\int \frac{\sin(\theta)\cos(\theta)}{3\cos(\theta)}\,d\theta$$

$$= 3\int \sin(\theta)\,d\theta$$

$$= 3\left[-\cos(\theta)\right] + C$$

$$= -3\cos(\theta) + C$$

$$= -3\left(\frac{\sqrt{9-x^2}}{3}\right) + C$$

$$\boxed{= -\sqrt{9-x^2} + C}$$

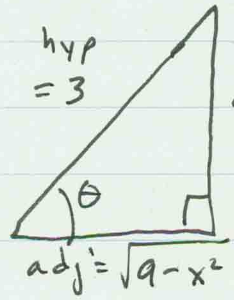
OR $\longrightarrow$

Key: ★★

$\sqrt{9-x^2} = \sqrt{a^2-u^2}$

$a^2 = 9$, $a = 3$

$u = x$



$\frac{x}{3} = \frac{\text{opp}}{\text{hyp}}$

Let $\frac{x}{3} = \sin(\theta)$

$x = 3\sin(\theta)$

$\frac{dx}{d\theta} = 3\cos(\theta)$

$dx = 3\cos(\theta)\,d\theta$

$\cos(\theta) = \frac{\text{adj}}{\text{hyp}}$

$\cos(\theta) = \frac{\sqrt{9-x^2}}{3}$

check! $\frac{d}{d(x)}\left[-\sqrt{9-x^2} + c\right]$

$= -\frac{d}{dx}\left[(9-x^2)^{1/2}\right] + 0$

$= -\left[\frac{1}{2}(9-x^2)^{-1/2} \cdot \frac{d}{dx}(9-x^2)\right]$

$= -\left[\frac{1}{2}(9-x^2)^{-1/2}\right](-2x)$

$= x\,(9-x^2)^{-1/2}$

$= \frac{x}{\sqrt{9-x^2}}$

8.4 Again . . . . "Cleaner"

22

$$\int \frac{x}{\sqrt{9-x^2}}\,dx$$

$$= \int \tan(\theta)\ (3\cos(\theta)\,d\theta)$$

$$= 3\int \frac{\sin(\theta)}{\cos(\theta)} \cdot \cos(\theta)\,d\theta$$

$$= 3\int \sin(\theta)\,d\theta$$

$$= 3\left[-\cos(\theta)\right] + C$$

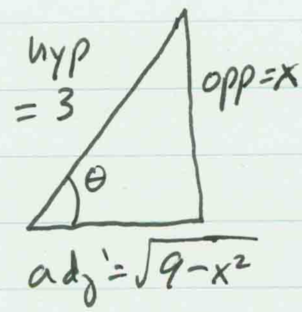
$$= -3\left(\frac{\sqrt{9-x^2}}{3}\right) + C$$

$$\boxed{= -\sqrt{9-x^2} + C}$$

Key: ☆☆

$\sqrt{9-x^2} = \sqrt{a^2-u^2}$

$a=3, \quad x=u$



$\frac{x}{3} = \frac{opp}{hyp}$

let $\frac{x}{3} = \sin(\theta)$

$x = 3\sin(\theta)$

$\frac{dx}{d\theta} = 3\cos(\theta)$

$dx = 3\cos(\theta)\,d\theta$

$\frac{x}{\sqrt{9-x^2}} = \frac{opp}{adj}$

$\frac{x}{\sqrt{9-x^2}} = \tan(\theta)$

$\cos(\theta) = \frac{adj}{hyp}$

$\cos(\theta) = \frac{\sqrt{9-x^2}}{3}$

$\boxed{8.4}$

#30

$$\int \frac{\sqrt{4x^2+9}}{x^4}\,dx$$

$$= \int \frac{3\sec(\theta)}{\left[\frac{3}{2}\tan(\theta)\right]^4}\left(\frac{3}{2}\sec^2(\theta)\,d\theta\right)$$

$$= \frac{9}{2}\int \frac{\sec^3(\theta)\,d\theta}{\frac{81}{16}\tan^4(\theta)}$$

$$= \frac{1}{\frac{9}{8}}\int \frac{\left[\frac{1}{\cos(\theta)}\right]^3}{\left[\frac{\sin(\theta)}{\cos(\theta)}\right]^4}\,d\theta$$

$$= \frac{8}{9}\int \frac{1}{\cos^3(\theta)}\cdot\frac{\cos^4(\theta)}{\sin^4(\theta)}\,d\theta$$

$$= \frac{8}{9}\int \frac{\cos(\theta)}{\sin^4(\theta)}\,d\theta$$

$$= \frac{8}{9}\int \frac{\cos(\theta)}{(z)^4}\left(\frac{dz}{\cos(\theta)}\right)$$

$$= \frac{8}{9}\int z^{-4}\,dz$$

$$= \frac{8}{9}\left[\frac{z^{-3}}{-3}\right] + c$$

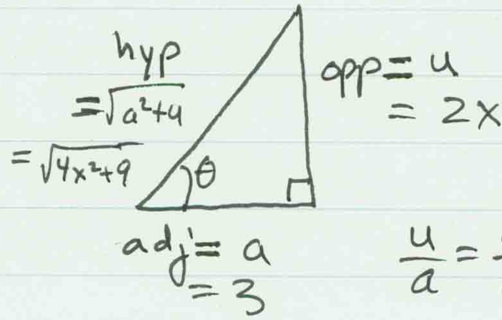
$$= \frac{8}{-27\cdot z^3} + c$$

$$= \frac{-8}{27\,(\sin(\theta))^3} + c$$

$$= \frac{-8}{27}\csc^3(\theta) + C$$

★★ Key: $\sqrt{a^2+u^2}$
$= \sqrt{4x^2+9}$

$u^2 = 4x^2$ | $9 = a^2$
$u = 2x$ | $3 = a$



$$\frac{u}{a} = \frac{opp}{adj}$$

$$\frac{2x}{3} = \tan\theta$$

$$2x = 3\tan(\theta)$$

$$x = \frac{3}{2}\tan(\theta)$$

$$\frac{dx}{d\theta} = \frac{3}{2}\cdot\sec^2(\theta)$$

$$dx = \frac{3}{2}\sec^2(\theta)\,d\theta$$

$$\frac{hyp}{adj} = \frac{\sqrt{4x^2+9}}{3} = \sec(\theta)$$

$$\sqrt{4x^2+9} = 3\sec(\theta)$$

Let $z = \sin(\theta)$

$$\frac{dz}{d\theta} = \cos(\theta)$$

$$\frac{dz}{\cos(\theta)} = d\theta$$

$$\csc(\theta) = \frac{hyp}{opp} = \frac{\sqrt{4x^2+9}}{2x}$$

$\boxed{8.4}$

#30 cont'd

$$= -\frac{8}{27}\left(\frac{\sqrt{4x^2+9}}{2x}\right)^3 + C$$

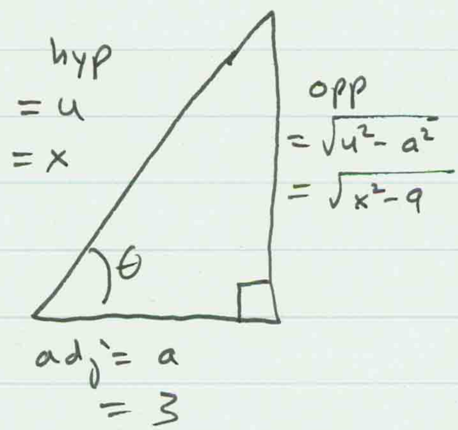
$$= -\frac{8}{27}\cdot\frac{(4x^2+9)^{3/2}}{8x^3} + C$$

$$\boxed{= -\frac{(4x^2+9)^{3/2}}{27x^3} + C}$$

Key: $\sqrt{u^2-a^2}$ ★★

$= \sqrt{x^2-9}$

$u = x$, $a = 3$



$\frac{u}{a} = \frac{x}{3} = \frac{hyp}{adj}$

$\frac{x}{3} = \sec(\theta)$

$x = 3\sec(\theta)$

$\frac{dx}{d\theta} = 3\sec(\theta)\tan(\theta)$

$dx = 3\sec(\theta)\tan(\theta)\,d\theta$

$\frac{\sqrt{x^2-9}}{x} = \frac{opp}{hyp} = \boxed{\sin\theta}$

#52

$$\int_3^6 \frac{\sqrt{x^2-9}}{x^2}\,dx$$

(b) change variables:

$x = 3\sec(\theta)$

$3 = 3\sec(\theta)$

$1 = \sec(\theta)$

$1 = \frac{1}{\cos(\theta)}$

$\cos(\theta) = 1$, $\theta = \cos^{-1}(1)$

$\underline{\theta = 0}$

$6 = 3\sec(\theta)$

$2 = \sec(\theta)$

$2 = \frac{1}{\cos(\theta)}$

$\cos(\theta) = \frac{1}{2}$, $\theta = \cos^{-1}(\frac{1}{2})$

$\underline{\theta = \pi/3}$

$\boxed{8.4}$

$$\sec\left(\frac{\pi}{3}\right) = \frac{1}{\cos(\pi/3)} = \frac{1}{1/2} = 2$$

$$\tan(\pi/3) = \frac{\sin(\pi/3)}{\cos(\pi/3)} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

$$\sec(0) = 1$$

$$\tan(0) = 0$$

52 cont'd

(b)

$$\int_3^6 \frac{\sqrt{x^2-9}}{x^2}\,dx = \int_{x=3}^{x=6} \left(\frac{\sqrt{x^2-9}}{x}\right)\left(\frac{1}{x}\right)dx$$

$$= \int_{\theta=0}^{\theta=\pi/3} (\sin\theta)\left(\frac{1}{3\sec(\theta)}\right)\left(3\sec(\theta)\tan(\theta)\,d\theta\right)$$

$$= \int_0^{\pi/3} \sin(\theta)\left(\frac{\sin(\theta)}{\cos(\theta)}\right)d\theta$$

use $\sin^2(\theta) = 1 - \cos^2(\theta)$

$$= \int_0^{\pi/3} \frac{\sin^2(\theta)}{\cos(\theta)}\,d\theta$$

$$= \int_0^{\pi/3} \frac{1-\cos^2(\theta)}{\cos(\theta)}\,d\theta = \int_0^{\pi/3} \frac{1}{\cos(\theta)}\,d\theta - \int_0^{\pi/3} \frac{\cos^2(\theta)}{\cos(\theta)}\,d\theta$$

$$= \int_0^{\pi/3} \sec(\theta)\,d\theta - \int_0^{\pi/3} \cos(\theta)\,d\theta$$

$$= \Big[\ln|\sec(\theta)+\tan(\theta)|\Big]_0^{\pi/3} - \Big[\sin(\theta)\Big]_0^{\pi/3}$$

$$= \left(\ln|\sec(\pi/3)+\tan(\pi/3)| - \ln|\sec(0)+\tan(0)|\right) - \left(\sin(\pi/3) - \sin(0)\right)$$

$$= \left(\ln|2+\sqrt{3}| - \ln|1+0|\right) - \left(\frac{\sqrt{3}}{2} - 0\right)$$

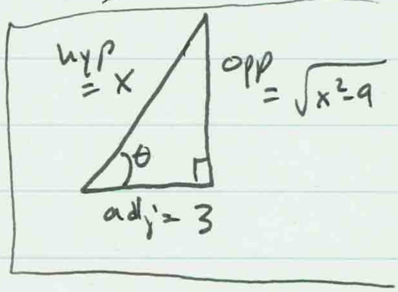
$$= \ln|2+\sqrt{3}| - \ln|1| - \frac{\sqrt{3}}{2}$$

(b) $\boxed{= \ln(2+\sqrt{3}) - \frac{\sqrt{3}}{2}}$

$\boxed{8.4}$

#52 cont'd

(a) $$\int_3^6 \frac{\sqrt{x^2-9}}{x^2}\,dx = \int_{x=3}^{x=6} (\sin\theta)\left(\frac{1}{3\sec(\theta)}\right)(3\sec(\theta)\tan(\theta)\,d\theta)$$



$$= \int_{x=3}^{x=6} \frac{\sin^2(\theta)}{\cos(\theta)}\,d\theta = \int_{x=3}^{x=6} \frac{1-\cos^2(\theta)}{\cos(\theta)}\,d\theta$$

$$= \int_{x=3}^{x=6} \left[\frac{1}{\cos(\theta)} - \frac{\cos^2(\theta)}{\cos(\theta)}\right] d\theta = \int_{x=3}^{x=6} (\sec(\theta) - \cos(\theta))\,d\theta$$

$$= \Big[\ln|\sec(\theta)+\tan(\theta)| - \sin(\theta)\Big]_{x=3}^{x=6}$$

$\tan(\theta) = \frac{opp}{adj}$

$\tan(\theta) = \frac{\sqrt{x^2-9}}{3}$

$$= \left[\ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| - \frac{\sqrt{x^2-9}}{x}\right]_{x=3}^{x=6}$$

$$= \left(\ln\left|\frac{(6)}{3} + \frac{\sqrt{(6)^2-9}}{3}\right| - \frac{\sqrt{(6)^2-9}}{(6)}\right) - \left(\ln\left|\frac{(3)}{3} + \frac{\sqrt{(3)^2-9}}{3}\right| - \frac{\sqrt{(3)^2-9}}{(3)}\right)$$

$$= \left(\ln\left|2 + \frac{\sqrt{36-9}}{3}\right| - \frac{\sqrt{36-9}}{6}\right) - \left(\ln\left|1 + \frac{\sqrt{9-9}}{3}\right| - \frac{\sqrt{9-9}}{3}\right)$$

$$= \left(\ln\left|2 + \frac{\sqrt{27}}{3}\right| - \frac{\sqrt{27}}{6}\right) - \left(\ln|1+0| - 0\right)$$

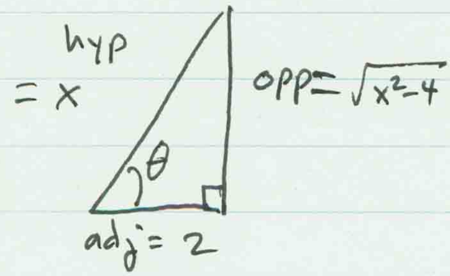
$$= \ln\left(2 + \frac{3\sqrt{3}}{3}\right) - \frac{3\sqrt{3}}{6} - \ln(1)$$

$$\boxed{= \ln(2+\sqrt{3}) - \frac{\sqrt{3}}{2}}$$

$\boxed{8.4}$

#10 $\displaystyle\int \frac{\sqrt{x^2-4}}{x}\,dx$

★★ Key $= \sqrt{u^2-a^2}$
$= \sqrt{x^2-4}$
$u = x, \quad a = 2$



$$\frac{hyp}{adj} = \frac{x}{2} = \sec(\theta)$$

$$x = 2\sec(\theta)$$

$$\frac{dx}{d\theta} = 2\sec(\theta)\tan(\theta)$$

$$dx = 2\sec(\theta)\tan(\theta)\,d\theta$$

$$\boxed{\frac{\sqrt{x^2-4}}{x} = \frac{opp}{hyp} = \sin(\theta)}$$

$$\int \frac{\sqrt{x^2-4}}{x}\,dx = \int (\sin(\theta))(2\sec(\theta)\tan(\theta)\,d\theta)$$

$$= 2\int \tan^2(\theta)\,d\theta$$

$$= 2\int (\sec^2(\theta)-1)\,d\theta$$

use $\tan^2(\theta) = \sec^2(\theta) - 1$

$$= 2\left[\int \sec^2(\theta)\,d\theta - \int 1\,d\theta\right]$$

$$= 2\left[\tan(\theta) - \theta\right] + C$$

convert to "x"

$$\tan(\theta) = \frac{opp}{adj} = \frac{\sqrt{x^2-4}}{2}$$

$$\frac{x}{2} = \sec(\theta)$$

$$\operatorname{arcsec}\left(\frac{x}{2}\right) = \operatorname{arcsec}(\sec(\theta))$$

$$\operatorname{arcsec}\left(\frac{x}{2}\right) = \theta$$

$$= 2\tan(\theta) - 2\theta + C$$

$$= 2\left(\frac{\sqrt{x^2-4}}{2}\right) - 2\left[\operatorname{arcsec}\left(\frac{x}{2}\right)\right] + C$$

$$\boxed{= \sqrt{x^2-4} - 2\operatorname{arcsec}\left(\frac{x}{2}\right) + C}$$