$$\boxed{\int \frac{1}{u}\,du = \ln|u| + c}$$

1/13

8.5 Partial Fractions

$$\frac{x}{x^2-5x+6} = \frac{A}{x-2} + \frac{B}{x-3}$$

x^2-5x+6
$= (x-2)(x-3)$

Distinct Linear Factors

$$(x-2)(x-3)\left[\frac{x}{x^2-5x+6}\right] = (x-2)(x-3)\left[\frac{A}{x-2} + \frac{B}{x-3}\right]$$

$$x = A(x-3) + B(x-2)$$

$$x = Ax - 3A + Bx - 2B$$

$$x = (Ax + Bx) + (-3A - 2B)$$

$$1x + 0 = (A+B)x + (-3A-2B)$$

Match coefficients

system of Equations
$$\begin{cases} 1 = A + B \\ 0 = -3A - 2B \end{cases}$$

$\longrightarrow$ Matrix $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -3 & -2 & 0 \end{array}\right]$

solve using TI-83

solve by Addition/Elimination

$$3(1) = 3(A+B)$$

$$\begin{aligned} 3 &= 3A + 3B \\ + \quad 0 &= -3A - 2B \\ \hline 3 &= B \end{aligned}$$

Find A: $1 = A + B$

$1 = A + (3)$

$-2 = A$

$$\boxed{\text{So, } \frac{x}{x^2-5x+6} = \frac{-2}{x-2} + \frac{3}{x-3}}$$

$\boxed{8.5}$

Irreducible Quadratic Factor

Recall: $x^3-1=(x-1)(x^2+x+1)$

$$\frac{3x-5}{x^3-1}=\frac{3x-5}{(x-1)(x^2+x+1)}$$

$$\frac{3x-5}{x^3-1}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}$$

$$(x-1)(x^2+x+1)\left(\frac{3x-5}{x^3-1}\right)=(x-1)(x^2+x+1)\left[\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}\right]$$

$$3x-5=A(x^2+x+1)+(Bx+C)(x-1)$$

choose $x=1$

$$3(1)-5=A((1)^2+(1)+1)+[B(1)+C][(1)-1]$$

$$3-5=A(3)+(B+C)\cdot 0$$

$$-2=3A$$

$$\boxed{-\frac{2}{3}=A}$$

So, $3x-5=-\frac{2}{3}(x^2+x+1)+(Bx+c)(x-1)$

$$3\cdot(3x-5)=3\left[-\tfrac{2}{3}(x^2+x+1)+(Bx+c)(x-1)\right]$$

$$9x-15=-2(x^2+x+1)+3(Bx+c)(x-1)$$

$$9x-15=-2x^2-2x-2+3[Bx^2+cx-Bx-c]$$

$$(2x^2+2x+2)+9x-15=(2x^2+2x+2)-2x^2-2x-2+3[Bx^2+(c-B)x-c]$$

$$2x^2+11x-13=3Bx^2+3(c-B)x-3c$$

match coefficients

$\boxed{8.5}$

System of Equations

$$\begin{cases} 2 = 3B \\ 11 = 3(C-B) \\ -13 = -3C \end{cases}$$

$$2 = 3B \qquad , \qquad -13 = -3C$$

$$\boxed{\frac{2}{3} = B} \qquad \frac{-13}{-3} = C$$

$$\boxed{\frac{13}{3} = C}$$

$$11 = 3[C-B]$$
$$11 = 3\left[\left(\frac{13}{3}\right) - \left(\frac{2}{3}\right)\right]$$
$$11 = 13 - 2$$
$$11 = 11 \text{ , True !}$$

So,

$$\frac{3x-5}{x^3-1} = \frac{3x-5}{(x-1)(x^2+x+1)}$$

$$= \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

$$\boxed{\frac{3x-5}{x^3-1} = \frac{-2/3}{x-1} + \frac{\frac{2}{3}x + \frac{13}{3}}{x^2+x+1}}$$

$\boxed{8.5}$

#8 $\int \frac{1}{4x^2-9}\,dx$

$$\frac{1}{4x^2-9} = \frac{1}{(2x+3)(2x-3)}$$

$$(2x+3)(2x-3)\left[\frac{1}{4x^2-9}\right] = \left[\frac{A}{2x+3} + \frac{B}{2x-3}\right](2x+3)(2x-3)$$

$$1 = A(2x-3) + B(2x+3)$$

choose $x = 3/2$

$$1 = A[2(3/2)-3] + B[2(3/2)+3]$$
$$1 = A[3-3] + B[3+3]$$
$$1 = A\cdot 0 + 6B$$
$$1 = 6B$$
$$\boxed{\tfrac{1}{6} = B}$$

choose $x = -3/2$

$$1 = A[2(-3/2)-3] + B[2(-3/2)+3]$$
$$1 = A(-3-3) + B(0)$$
$$1 = -6A$$
$$\boxed{\frac{-1}{6} = A}$$

So, $\frac{1}{4x^2-9} = \frac{-\frac{1}{6}}{2x+3} + \frac{\frac{1}{6}}{2x-3}$

8.5

#8 con't'd

$$\int \frac{1}{4x^2-9}\,dx = \int \left[\frac{-\frac{1}{6}}{2x+3} + \frac{\frac{1}{6}}{2x-3} \right] dx$$

$$= -\frac{1}{6}\int \frac{1}{2x+3}\,dx + \frac{1}{6}\int \frac{1}{2x-3}\,dx$$

let $u = 2x+3$	let $z = 2x-3$
$\frac{du}{dx} = 2$	$\frac{dz}{dx} = 2$
$\frac{du}{2} = dx$	$\frac{dz}{2} = dx$

$$= -\frac{1}{6}\int \frac{1}{u}\left(\frac{du}{2}\right) + \frac{1}{6}\int \frac{1}{z}\left(\frac{dz}{2}\right)$$

$$= -\frac{1}{12}\int \frac{1}{u}\,du + \frac{1}{12}\int \frac{1}{z}\,dz$$

$$= -\frac{1}{12}\cdot \ln|u| + \frac{1}{12}\cdot \ln|z| + C$$

$$= -\frac{1}{12}\ln|2x+3| + \frac{1}{12}\ln|2x-3| + C$$

$$= \frac{1}{12}\ln|2x-3| - \frac{1}{12}\ln|2x+3| + C$$

$$= \frac{1}{12}\left[\ln|2x-3| - \ln|2x+3|\right] + C$$

$$\boxed{= \frac{1}{12}\cdot \ln\left|\frac{2x-3}{2x+3}\right| + C}$$

14 $\displaystyle\int \frac{x^3-x+3}{x^2+x-2}\,dx$

1st $\boxed{\text{Division}}$

$$
\begin{array}{r|l}
 & \qquad\qquad x-1 \\
x^2+x-2 & \overline{\,x^3+0x^2-x+3\,} \\
 & -x^3-x^2+2x \\
\hline
 & \quad -x^2+x+3 \\
 & \quad +x^2+x-2 \\
\hline
 & \qquad 2x+1 \quad \leftarrow \text{Remainder}
\end{array}
$$

x^3+x-2x ← change signs in subtraction

$-x^2-x+2$ ← change signs

So

$$\frac{x^3-x+3}{x^2+x-2} = x-1 + \boxed{\frac{2x+1}{x^2+x-2}} \leftarrow \text{use Partial Fraction Method}$$

$$\frac{2x+1}{x^2+x-2} = \frac{2x+1}{(x+2)(x-1)}$$

$$(x+2)(x-1)\cdot\left[\frac{2x+1}{x^2+x-2}\right] = \left[\frac{A}{x+2} + \frac{B}{x-1}\right](x+2)(x-1)$$

$$2x+1 = A(x-1) + B(x+2)$$

choose $x=1$

$$2(1)+1 = A[(1)-1] + B[(1)+2]$$

$$3 = A\cdot 0 + 3B$$

$$3 = 3B$$

$$\frac{3}{3} = B$$

$$\boxed{1 = B}$$

#14 cont'd

choose $x = -2$

$$2(-2)+1 = A[(-2)-1] + B[(-2)+2]$$
$$-4+1 = -3A + B \cdot 0$$
$$-3 = -3A$$
$$\frac{-3}{-3} = A$$
$$\boxed{1 = A}$$

This gives us $\dfrac{2x+1}{x^2+x-2} = \dfrac{1}{x+2} + \dfrac{1}{x-1}$

AND

$$\frac{x^3-x+3}{x^2+x-2} = x-1 + \frac{1}{x+2} + \frac{1}{x-1}$$

$$\int \frac{x^3-x+3}{x^2+x-2}\,dx = \int \left(x-1 + \frac{1}{x+2} + \frac{1}{x-1}\right)dx$$

$$= \int x\,dx - \int 1\,dx + \int \frac{1}{x+2}\,dx + \int \frac{1}{x-1}\,dx \quad ☆☆$$

$$\boxed{= \frac{x^2}{2} - x + \ln|x+2| + \ln|x-1| + C}$$

or

$$\boxed{= \frac{x^2}{2} - x + \ln|x^2+x-2| + C}$$ ✓

☆☆ let $u = x+2$, $\frac{du}{dx} = 1$, $du = dx$ | let $z = x-1$, $\frac{dz}{dx} = 1$, $dz = dx$

$\boxed{8.5}$

#20 $\int \frac{4x^2}{x^3+x^2-x-1}\,dx$

$$\frac{4x^2}{x^3+x^2-x-1} = \frac{4x^2}{x^2(x+1)-1(x+1)} = \frac{4x^2}{(x+1)(x^2-1)} = \frac{4x^2}{(x+1)(x+1)(x-1)}$$

$$(x+1)^2(x-1)\left[\frac{4x^2}{(x+1)^2(x-1)}\right] = \left[\frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}\right](x+1)^2(x-1)$$

$$4x^2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

choose $x=1$.

$$4(1)^2 = A[(1)+1]^2 + B[(1)-1][(1)+1] + C[(1)-1]$$

$$4 = A(2)^2 + B(0)(2) + C\cdot(0)$$

$$4 = 4A$$

$$\frac{4}{4} = A$$

$$\boxed{1 = A}$$

choose $x=-1$.

$$4(-1)^2 = 1[(-1)+1]^2 + B[(-1)-1][(-1)+1] + C[(-1)-1]$$

$$4 = 1(0)^2 + B(-2)(0) + C(-2)$$

$$4 = -2C$$

$$\frac{4}{-2} = C$$

$$\boxed{-2 = C}$$

$\boxed{8.5}$ #20 cont'd

choose $x=0$

$$4(0)^2 = 1((0)+1)^2 + B[(0)-1][(0)+1] - 2((0)-1)$$
$$0 = 1(1) + B(-1)(1) - 2(-1)$$
$$0 = 1 - B + 2$$
$$0 = 3 - B$$
$$\boxed{B = 3}$$

So, we have

$$\frac{4x^2}{x^3+x^2-x-1} = \frac{1}{x-1} + \frac{3}{x+1} + \frac{-2}{(x+1)^2}$$

AND

$$\int \frac{4x^2}{x^3+x^2-x-1}\,dx = \int \left(\frac{1}{x-1} + \frac{3}{x+1} - \frac{2}{(x+1)^2}\right)dx$$

$$= \int \frac{1}{x-1}\,dx + 3\int \frac{1}{x+1}\,dx - 2\int \frac{1}{(x+1)^2}\,dx$$

$$= \ln|x-1| + 3\cdot\ln|x+1| - 2\int u^{-2}(du)$$

$$= \ln|x-1| + 3\ln|x+1| - 2\left[\frac{u^{-1}}{-1}\right] + C$$

let $u = x+1$
$\frac{du}{dx} = 1$
$du = dx$

$$= \ln|x-1| + 3\ln|x+1| + 2(x+1)^{-1} + C$$

$$\boxed{= \ln|x-1| + 3\ln|x+1| + \frac{2}{x+1} + C}$$

8.5

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$$\int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)}\,dx$$

$$\frac{2x^3-4x-8}{(x^2-x)(x^2+4)} = \frac{2x^3-4x-8}{x(x-1)(x^2+4)}$$

$$x(x-1)(x^2+4)\left[\frac{2x^3-4x-8}{x(x-1)(x^2+4)}\right] = \left[\frac{A}{x}+\frac{B}{x-1}+\frac{Cx+D}{x^2+4}\right]x(x-1)(x^2+4)$$

Irreducible quadratic Factor

$$2x^3-4x-8 = A(x-1)(x^2+4) + Bx(x^2+4) + (Cx+D)x(x-1)$$

$$2x^3-4x-8 = A[x^3-x^2+4x-4] + Bx^3+4Bx + (Cx+D)(x^2-x)$$

$$2x^3-4x-8 = Ax^3-Ax^2+4Ax-4A+Bx^3+4Bx+Cx^3-Cx^2+Dx^2-Dx$$

$$2x^3+0\cdot x^2-4x-8 = (Ax^3+Bx^3+Cx^3)$$
$$+(-Ax^2-Cx^2+Dx^2)$$
$$+(4Ax+4Bx-Dx)$$
$$+(-4A)$$

Math Coefficients

System of Equations

$$\begin{cases} 2 = A+B+C \\ 0 = -A-C+D \\ -4 = 4A+4B-D \\ -8 = -4A \end{cases} \Rightarrow$$

MATRIX

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 2 \\ -1 & 0 & -1 & 1 & 0 \\ 4 & 4 & 0 & -1 & -4 \\ -4 & 0 & 0 & 0 & -8 \end{array}\right]$$

solve on TI-83

rref( )

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

$A=2$

$B=-2$

$C=2$

$D=4$

$\boxed{8.5}$

So, we have

$$\frac{2x^3-4x-8}{(x^2-x)(x^2+4)} = \frac{2}{x} + \frac{-2}{x-1} + \frac{2x+4}{x^2+4}$$

$$\int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)}\,dx = \int \left(\frac{2}{x} - \frac{2}{x-1} + \frac{2x+4}{x^2+4}\right)dx$$

$$= 2\int \frac{1}{x}\,dx - 2\int \frac{1}{x-1}\,dx + \int \frac{2x}{x^2+4}\,dx + 4\int \frac{1}{x^2+4}\,dx$$

$$= 2\ln|x| - 2\ln|x-1| + \int \frac{2x}{u}\left(\frac{du}{2x}\right) + 4\int \frac{1}{x^2+a^2}\,dx$$

$$\boxed{\begin{aligned} &\text{let } u = x^2+4 \\ &\frac{du}{dx} = 2x \\ &\frac{du}{2x} = dx \end{aligned}}$$

let $a^2 = 4$

$a = 2$

$$\boxed{\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c}$$

$$= 2\ln|x| - 2\ln|x-1| + \int \frac{1}{u}\,du + 4\left[\frac{1}{a}\arctan\left(\frac{x}{a}\right)\right] + c$$

$$= 2\ln|x| - 2\ln|x-1| + \ln|u| + 4\left[\frac{1}{2}\arctan\left(\frac{x}{2}\right)\right] + c$$

$$\boxed{= 2\ln|x| - 2\ln|x-1| + \ln|x^2+4| + 2\arctan\left(\frac{x}{2}\right) + c}$$

check???

$\boxed{8.5}$

#28

$$\int \frac{x^2+x+3}{x^4+6x^2+9}\,dx$$

$$\frac{x^2+x+3}{x^4+6x^2+9} = \frac{x^2+x+3}{(x^2+3)(x^2+3)} = \frac{x^2+x+3}{(x^2+3)^2}$$

Repeated Quadratic Factors

$$(x^2+3)^2 \cdot \left[\frac{x^2+x+3}{(x^2+3)^2}\right] = \left[\frac{Ax+B}{(x^2+3)} + \frac{Cx+D}{(x^2+3)^2}\right] \cdot (x^2+3)^2$$

$$x^2+x+3 = (Ax+B)(x^2+3) + Cx+D$$

$$x^2+x+3 = Ax^3+Bx^2+3Ax+3B+Cx+D$$

$$0x^3+x^2+x+3 = Ax^3+Bx^2+(3A+C)x+(3B+D)$$

Match Coefficients

system of Equations

$$\begin{cases} 0 = A \leftarrow \star \\ 1 = B \leftarrow \star \\ 1 = 3A+C \\ 3 = 3B+D \end{cases}$$

$\Rightarrow$

$$1 = 3A+C$$
$$1 = 3(0)+C$$
$$\boxed{1 = C}$$

$$3 = 3B+D$$
$$3 = 3(1)+D$$
$$3 = 3+D$$
$$\boxed{0 = D}$$

8.5 #28 Cont'd

So, we have

$$\frac{x^2+x+3}{x^4+6x^2+9} = \frac{1}{x^2+3} + \frac{x}{(x^2+3)^2}$$

AND

$$\int \frac{x^2+x+3}{x^4+6x^2+9}\,dx = \int\left(\frac{1}{x^2+3} + \frac{x}{(x^2+3)^2}\right)dx$$

$$= \int \frac{1}{x^2+3}\,dx + \int \frac{x}{(x^2+3)^2}\,dx$$

let $a^2 = 3$
$a = \sqrt{3}$

let $u = x^2+3$
$\frac{du}{dx} = 2x$
$\frac{du}{2x} = dx$

$$= \int \frac{1}{x^2+a^2}\,dx + \int \frac{x}{(u)^2}\left(\frac{du}{2x}\right)$$

$$= \frac{1}{a}\arctan\left(\frac{x}{a}\right) + \frac{1}{2}\int u^{-2}\,du$$

$$= \frac{1}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) + \frac{1}{2}\left(\frac{u^{-1}}{-1}\right) + C$$

$$= \frac{1}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) - \frac{1}{2u} + C$$

$$\boxed{= \frac{1}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) - \frac{1}{2(x^2+3)} + C}$$