8.6 Integration by Tables

#4

$$\int \frac{\sqrt{x^2-9}}{3x}\,dx$$

, use Integration Formula

$$\int \frac{\sqrt{u^2-a^2}}{u}\,du = \sqrt{u^2-a^2} - a\,\mathrm{sec}\,\frac{|u|}{a} + C$$

let $u = x$ & $a^2 = 9$

$\frac{du}{dx} = 1$ $a = 3$

$du = dx$

$$\int \frac{\sqrt{x^2-9}}{3x}\,dx = \frac{1}{3}\int \frac{\sqrt{u^2-a^2}}{u}\,du$$

$$= \frac{1}{3}\left[\sqrt{u^2-a^2} - a\,\mathrm{sec}\,\frac{|u|}{a}\right] + C$$

$$\boxed{= \frac{1}{3}\left[\sqrt{x^2-9} - 3\,\mathrm{sec}\,\frac{|x|}{3}\right] + C}$$

$\boxed{8.6}$

#9

$$\int \frac{1}{\sqrt{x}\left[1-\cos(\sqrt{x})\right]}\,dx$$

Integration Formula

$$\boxed{\int \frac{1}{1 \pm \cos(u)}\,du = -\cot(u) \pm \csc(u) + C}$$

$$\boxed{\begin{array}{ll} \text{let } u = \sqrt{x} & \underline{2\sqrt{x}\cdot du = dx} \\ \quad u = x^{1/2} & \\ \quad \frac{du}{dx} = \frac{1}{2}x^{-1/2} & \\ \quad \frac{du}{dx} = \frac{1}{2\sqrt{x}} & \end{array}}$$

$$\int \frac{1}{\sqrt{x}\left[1-\cos(\sqrt{x})\right]}\,dx = \int \frac{1}{\sqrt{x}\left[1-\cos(u)\right]}\left(2\sqrt{x}\,du\right)$$

$$= 2\int \frac{1}{1-\cos(u)}\,du$$

$$= 2\left[-\cot(u) - \csc(u)\right] + C$$

$$= 2\left[-\cot(\sqrt{x}) - \csc(\sqrt{x})\right] + C$$

$$\boxed{= -2\cot\sqrt{x} - 2\csc\sqrt{x} + C}$$

$\boxed{8.6}$

#12 $\int e^{-\frac{x}{2}} \sin(2x)\,dx$

Integration Formula

$$\boxed{\int e^{au} \sin(bu)\,du = \frac{e^{au}}{a^2+b^2}\left[a\sin(bu) - b\cos(bu)\right] + C}$$

let $u = x$	$a = -1/2$
$\frac{du}{dx} = 1$	$b = 2$
$du = dx$	

$$\int e^{-x/2} \sin(2x)\,dx = \int e^{au} \sin(bu)\,du$$

$$= \frac{e^{au}}{a^2+b^2}\left[a\sin(bu) - b\cos(bu)\right] + C$$

$$= \frac{e^{-\frac{1}{2}x}}{(-1/2)^2 + (2)^2}\left[-\frac{1}{2}\sin(2x) - 2\cos(2x)\right] + C$$

$$= \frac{e^{-1/2x}}{\frac{1}{4} + \frac{4}{1}\cdot\frac{4}{4}}\left[-\frac{1}{2}\sin(2x) - 2\cos(2x)\right] + C$$

$$= \frac{e^{-\frac{x}{2}}}{\frac{17}{4}}\left[-\frac{1}{2}\sin(2x) - 2\cos(2x)\right] + C$$

$$\boxed{= -\frac{4}{17} e^{-\frac{x}{2}}\left[\frac{1}{2}\sin(2x) + 2\cos(2x)\right] + C}$$

$\boxed{8.6}$

#17

$$\int \frac{1}{x^2(x+1)}\,dx$$

Integration Formula

$$\boxed{\int \frac{1}{u^2(a+bu)}\,du = \frac{-1}{a}\left[\frac{1}{u} + \frac{b}{a}\ln\left|\frac{u}{a+bu}\right|\right] + C}$$

Let $u = x$	$a = 1$
$\frac{du}{dx} = 1$	$b = 1$
$du = dx$	

(a)

$$\int \frac{1}{x^2(x+1)}\,dx = \int \frac{1}{u^2(u+1)}\,du$$

$$= -\frac{1}{a}\left[\frac{1}{u} + \frac{b}{a}\ln\left|\frac{u}{a+bu}\right|\right] + C$$

$$= \frac{-1}{(1)}\left[\frac{1}{(x)} + \frac{(1)}{(1)}\cdot\ln\left|\frac{(x)}{(1)+(1)(x)}\right|\right] + C$$

$$\boxed{= -\left[\frac{1}{x} + \ln\left|\frac{x}{1+x}\right|\right] + C}$$

(b) Partial Fractions.

$$x^2(x+1)\cdot\left[\frac{1}{x^2(x+1)}\right] = \left[\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}\right]\cdot x^2(x+1)$$

$$1 = A(x)(x+1) + B(x+1) + Cx^2$$

let $x=0$

$$1 = A(0)[(0)+1] + B[(0)+1] + C(0)^2$$

$$\boxed{1 = B}$$

let $x=-1$

$$1 = A(-1)[(-1)+1] + 1\cdot[(-1)+1] + C(-1)^2$$

$$1 = A(-1)[0] + 1\cdot(0) + C\cdot 1$$

$$\boxed{1 = C}$$

let $x=1$

$$1 = A(1)[(1)+1] + 1\cdot[(1)+1] + 1(1)^2$$

$$1 = A\cdot 2 + 2 + 1$$

$$1 = 2A + 3$$

$$-2 = 2A$$

$$\boxed{-1 = A}$$

So

$$\boxed{\frac{1}{x^2(x+1)} = \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1}}$$

AND

$$\int \frac{1}{x^2(x+1)}\,dx = \int\left(\frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1}\right)dx$$

$$= -\int \frac{1}{x}\,dx + \int \frac{1}{x^2}\,dx + \int \frac{1}{x+1}\,dx$$

$$= -\ln|x| + \int x^{-2}\,dx + \ln|x+1|$$

$$= -\ln|x| + \ln|x+1| + (-1\cdot x^{-1}) + C$$

$$\boxed{= -\ln|x| + \ln|x+1| - \frac{1}{x} + C}$$

$\boxed{8.6}$

#26

$$\int \frac{e^x}{1-\tan(e^x)}\,dx$$

Integration Formula

$$\boxed{\int \frac{1}{1 \pm \tan(u)}\,du = \frac{1}{2}\left[u \pm \ln|\cos(u) \pm \sin(u)|\right] + C}$$

$$\boxed{\begin{aligned} \text{Let } u &= e^x \\ \frac{du}{dx} &= e^x \\ \frac{du}{e^x} &= dx \end{aligned}}$$

$$\int \frac{e^x}{1-\tan(e^x)}\,dx = \int \frac{e^x}{1-\tan(u)} \cdot \left(\frac{du}{e^x}\right)$$

$$= \int \frac{1}{1-\tan(u)}\,du$$

$$= \frac{1}{2}\left[u - \ln|\cos(u) - \sin(u)\right] + C$$

$$\boxed{= \frac{1}{2}\left[e^x - \ln|\cos(e^x) - \sin(e^x)|\right] + C}$$

$\boxed{8.6}$

#64

$$\int \frac{\sin(\theta)}{1+\cos^2(\theta)}\,d\theta$$

Integration formula

$$\boxed{\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C}$$

let $u = \cos(\theta)$	$a^2 = 1$
$\frac{du}{d\theta} = -\sin(\theta)$	$a = 1$
$\frac{du}{-\sin(\theta)} = d\theta$	

$$\int \frac{\sin(\theta)}{1+\cos^2(\theta)}\,d\theta = \int \frac{\sin(\theta)}{a^2+u^2}\left(\frac{du}{-\sin(\theta)}\right)$$

$$= -\int \frac{1}{a^2+u^2}\,du$$

$$= -\left[\frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)\right] + C$$

$$= -\left[\frac{1}{(1)}\tan^{-1}\left(\frac{\cos(\theta)}{1}\right)\right] + C$$

$$\boxed{= -\tan^{-1}[\cos(\theta)] + C}$$

$\sin^2(\theta) + \cos^2(\theta) = 1$

$1 + \cot^2(\theta) = \csc^2(\theta)$

$\boxed{8.6}$

#68

$$\int \frac{\cos(\theta)}{1+\cos(\theta)}\,d\theta = \int \left(\left[\frac{\cos(\theta)}{1+\cos(\theta)}\right]\cdot\left[\frac{1-\cos(\theta)}{1-\cos(\theta)}\right]\right)d\theta$$

↑ cool conjugates

$$= \int\left[\frac{\cos(\theta)-\cos^2(\theta)}{1-\cos^2(\theta)}\right]d\theta$$

$$= \int \frac{\cos(\theta)}{\sin^2(\theta)}\,d\theta - \int \frac{\cos^2(\theta)}{\sin^2(\theta)}\,d\theta$$

$$= \int\left[\frac{\cos(\theta)}{\sin(\theta)}\right]\cdot\left[\frac{1}{\sin(\theta)}\right]d\theta - \int \cot^2(\theta)\,d\theta$$

$$= \int \cot(\theta)\csc(\theta)\,d\theta - \int(\csc^2(\theta)-1)\,d\theta$$

$$= -\csc(\theta) - (-\cot(\theta) - \theta) + C$$

$$\boxed{= -\csc(\theta) + \cot(\theta) + \theta + C}$$

☆ Consider $u = \tan\left(\frac{x}{2}\right)$

$\tan^{-1}(u) = \tan^{-1}\left[\tan\left(\frac{x}{2}\right)\right]$

$\tan^{-1}(u) = x/2$

$2\tan^{-1}(u) = x$

$\frac{d}{du}\left[2\tan^{-1}(u)\right] = \frac{d}{du}(x)$

$2\cdot\frac{d}{du}\left[\tan^{-1}(u)\right] = \frac{dx}{du}$

$2\cdot\left[\frac{1}{1+u^2}\right] = \frac{dx}{du}$

$$\boxed{\frac{2\,du}{1+u^2} = dx}$$

"check this → out" →

$\boxed{8.6}$ Substitution for Rational Functions of Sine & Cosine

$$\boxed{\begin{array}{l} u = \dfrac{\sin(x)}{1+\cos(x)} = \tan\left(\frac{x}{2}\right) \\ \cos(x) = \dfrac{1-u^2}{1+u^2}, \quad \sin(x) = \dfrac{2u}{1+u^2}, \; \& \; dx = \dfrac{2\,du}{1+u^2} \end{array}}$$

#65
$$\int_0^{\pi/2} \frac{1}{1+\sin(\theta)+\cos(\theta)}\,d\theta = \int_{u=0}^{u=1} \left[\frac{1}{1+\left(\frac{2u}{1+u^2}\right)+\left(\frac{1-u^2}{1+u^2}\right)}\right]\cdot\left(\frac{2\,du}{1+u^2}\right)\left[\frac{1+u^2}{1}\right]$$

If			
$\theta = \frac{\pi}{2}$	$\theta = 0$	$\cos(\theta) = \frac{1-u^2}{1+u^2}$	$d\theta = \frac{2\,du}{1+u^2}$
$u = \tan\left(\frac{\pi/2}{2}\right)$	$u = \tan\left(\frac{0}{2}\right)$	$\sin(\theta) = \frac{2u}{1+u^2}$	
$u = \tan\left(\frac{\pi}{4}\right)$	$u = \tan(0)$		
$u = 1$	$u = 0$		

$$= 2\int_0^1 \frac{1}{(1+u^2)+(2u)+(1-u^2)}\,du$$

$$= 2\int_0^1 \frac{1}{2+2u}\,du = \frac{2}{2}\int_0^1 \frac{1}{1+u}\,du$$

$$= \int_0^1 \frac{1}{1+u}\,du$$

$$= \Big[\ln|1+u|\Big]_0^1$$

$$= \ln|1+(1)| - \ln|1+(0)|$$

$$= \ln(2) - \ln(1) = \boxed{\ln(2)}$$