$\boxed{8.7}$ L'Hopital's Rule:

Let f & g be differentiable functions on (a,b), an open interval containing c, except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a,b), except possibly at c itself.

If the limit of $\frac{f(x)}{g(x)}$ as x aproaches c produces the indeterminate form $\boxed{\frac{0}{0}}$ OR $\boxed{\frac{\infty}{\infty}}$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

Also, $\displaystyle \lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$

Main Indeterminate Forms: $\frac{0}{0}$ & $\frac{\infty}{\infty}$

Example:

$$\lim_{x \to 2} \frac{\sin(x-2)}{2x-4}$$

Stop!

$$\lim_{x \to 2} \frac{\sin(x-2)}{2x-4} = \frac{\sin(2-2)}{2(2)-4} = \frac{\sin(0)}{0} = \boxed{\frac{0}{0}}$$

$$\lim_{x \to 2} \frac{\sin(x-2)}{2x-4} = \lim_{x \to 2} \frac{\frac{d}{dx}[\sin(x-2)]}{\frac{d}{dx}(2x-4)}$$

$$= \lim_{x \to 2} \frac{\cos(x-2) \cdot 1}{2}$$

$$= \frac{\cos((2)-2)}{2} = \frac{\cos(0)}{2}$$

$$= \frac{1}{2} \checkmark$$

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Example:

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \frac{\sin(0)}{(0)} = \boxed{\frac{0}{0}}$$

stop!

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{\frac{d}{dx}[\sin(x)]}{\frac{d}{dx}[x]}$$

$$= \lim_{x\to 0} \frac{\cos(x)}{1}$$

$$= \cos(0)$$

$$= 1 \checkmark$$

#14

$$\lim_{x\to 2^-} \frac{\sqrt{4-x^2}}{x-2} = \frac{\sqrt{4-(2)^2}}{(2)-2} = \boxed{\frac{0}{0}}$$

stop!

$$\lim_{x\to 2^-} \frac{(4-x^2)^{1/2}}{x-2} = \lim_{x\to 2^-} \frac{\frac{d}{dx}\left[(4-x^2)^{1/2}\right]}{\frac{d}{dx}(x-2)}$$

$$= \lim_{x\to 2^-} \frac{\frac{1}{2}(4-x^2)^{-1/2}\cdot(-2x)}{(1)}$$

$$= \lim_{x\to 2^-} \frac{-x}{\sqrt{4-x^2}}$$

Not an Indeterminate Form

$$\text{“}\frac{-2}{0}\text{”}$$

$$= -\infty$$

"graph decreases without bound as x approaches 2 from the left"

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#34 $\displaystyle \lim_{x \to \infty} \frac{\ln(x^4)}{x^3} = \frac{\ln(\infty)}{(\infty)^3} = \boxed{\frac{\infty}{\infty}}$ Stop!

$$\lim_{x \to \infty} \frac{\ln(x^4)}{x^3} = \lim_{x \to \infty} \frac{\frac{d}{dx}[4\ln(x)]}{\frac{d}{dx}[x^3]}$$

$$= \lim_{x \to \infty} \frac{4 \cdot \left(\frac{1}{x}\right)}{3x^2}$$

$$= \lim_{x \to \infty} \frac{4}{3x^3}$$

$$\boxed{= 0}$$

WARNING!!

Notice!!

If you have $\frac{0}{0}$ or $\frac{\infty}{\infty}$ or $-\frac{\infty}{\infty}$ you can use L'Hopital's Rule.

If you have $\infty \cdot 0$, $\infty - \infty$, 0^0, 1^∞, or ∞^0, you we need to do/use a conversion technique until you get $\frac{0}{0}$, $\frac{\infty}{\infty}$, or $-\frac{\infty}{\infty}$.

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#40 $\displaystyle\lim_{x\to\infty} x\cdot\tan\left(\frac{1}{x}\right) = \infty\cdot\tan\left(\frac{1}{\infty}\right) = \boxed{\infty\cdot 0}$

Stop!

Needs to be $\boxed{\frac{0}{0}}$??

$$= \lim_{x\to\infty}\left[\frac{x\cdot\tan\left(\frac{1}{x}\right)}{1}\right]\cdot\left[\frac{\frac{1}{x}}{\frac{1}{x}}\right]$$

$$= \lim_{x\to\infty}\frac{\tan\left(\frac{1}{x}\right)}{\frac{1}{x}}$$

Now, we can use L'Hopital's Rule

$$= \lim_{x\to\infty}\frac{\frac{d}{dx}\left[\tan(x^{-1})\right]}{\frac{d}{dx}\left[x^{-1}\right]}$$

$$= \lim_{x\to\infty}\frac{\sec^2(x^{-1})\cdot\cancel{(-1\cdot x^{-2})}}{\cancel{(-1\cdot x^{-2})}}$$

$$= \lim_{x\to\infty}\sec^2\left(\frac{1}{x}\right)$$

$$= \sec^2\left[\lim_{x\to\infty}\frac{1}{x}\right]$$

★★★ Limits pass through composition of continuous functions!

$$= \sec^2(0)$$

$$\boxed{= 1}$$

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Example

$$\lim_{x\to 0}\left[\frac{1}{x\sin(x)}-\frac{1}{x^2}\right] = \frac{1}{0}-\frac{1}{0} = \boxed{\infty-\infty}$$ Stop!

$$\lim_{x\to 0}\left[\frac{1}{x\sin(x)}-\frac{1}{x^2}\right] = \lim_{x\to 0}\left[\frac{x}{x}\cdot\frac{1}{x\sin(x)}-\frac{\sin(x)}{\sin(x)}\cdot\frac{1}{x^2}\right]$$

stop!

$$= \lim_{x\to 0}\left[\frac{x-\sin(x)}{x^2\sin(x)}\right] = \frac{0-0}{0} = \boxed{\frac{0}{0}}$$

$$= \lim_{x\to 0}\frac{\frac{d}{dx}[x-\sin(x)]}{\frac{d}{dx}[x^2\sin(x)]}$$

$$= \lim_{x\to 0}\frac{1-\cos(x)}{x^2(\cos(x))+(\sin(x))(2x)}$$

stop!

$$= \lim_{x\to 0}\frac{1-\cos(x)}{x^2\cos(x)+2x\sin(x)} = \frac{1-1}{0+0} = \boxed{\frac{0}{0}}$$

$$= \lim_{x\to 0}\frac{\frac{d}{dx}[1-\cos(x)]}{\frac{d}{dx}[x^2\cos(x)+2x\sin(x)]}$$

$$= \lim_{x\to 0}\frac{-(-\sin x)}{x^2(-\sin(x))+2x\cos(x)+2x\cos(x)+2\sin(x)}$$

Stop!

$$= \lim_{x\to 0}\frac{\sin(x)}{-x^2\sin(x)+4x\cos(x)+2\sin(x)} = \frac{0}{0+0+0} = \boxed{\frac{0}{0}}$$

$$= \lim_{x\to 0}\frac{\frac{d}{dx}[\sin(x)]}{\frac{d}{dx}[-x^2\sin(x)+4x\cos(x)+2\sin(x)]}$$

$\boxed{8.7}$ $\infty - \infty$

Example cont'd

$$= \lim_{x \to 0} \frac{\cos(x)}{-x^2\cos(x) + \sin(x)(-2x) + 4x(-\sin(x)) + 4\cos(x) + 2\cos(x)}$$

$$= \lim_{x \to 0} \frac{\cos(x)}{-x^2\cos(x) - 6x\sin(x) + 6\cos(x)}$$

$$= \frac{\cos(0)}{0 + 0 + 6\cos(0)}$$

$$\boxed{= \frac{1}{6}}$$

#50

$$\lim_{x \to 0^+} \left[\cos\left(\tfrac{\pi}{2} - x\right)\right]^x$$

Identity

$\cos(\pi/2 - x) = \sin(x)$

$\sin(\frac{\pi}{2} - x) = \cos(x)$

stop!

$$= \lim_{x \to 0^+} [\sin(x)]^x = \boxed{0^0}$$

Let $y = \lim_{x \to 0^+} [\sin(x)]^x$

$$\ln(y) = \ln\left(\lim_{x \to 0^+} [\sin(x)]^x\right)$$

$$\ln(y) = \lim_{x \to 0^+} \left(\ln\{[\sin(x)]^x\}\right)$$

$$\ln(y) = \lim_{x \to 0^+} \left(x \cdot \ln[\sin(x)]\right)$$

$$\ln(y) = \lim_{x \to 0^+} \left(\frac{x \cdot \ln[\sin(x)]}{1} \cdot \boxed{\frac{\frac{1}{x}}{\frac{1}{x}}}\right)$$

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#50 cont'd

$$\ln(y) = \lim_{x \to 0^+} \left(\frac{\ln[\sin(x)]}{\frac{1}{x}} \right)$$

$$\ln(y) = \lim_{x \to 0^+} \left\{ \frac{\frac{d}{dx}\left[\ln[\sin(x)]\right]}{\frac{d}{dx}\left(x^{-1}\right)} \right\}$$

$$\ln(y) = \lim_{x \to 0^+} \frac{\frac{1}{\sin(x)} \cdot \cos(x)}{-1 \cdot x^{-2}}$$

can we do it ???

$$\ln(y) = \lim_{x \to 0^+} \frac{\cot(x)}{-\frac{1}{x^2}} \Longrightarrow \boxed{\frac{0}{0}}$$

Stop! Now, use l'Hopital's Rule

$$\ln(y) = \lim_{x \to 0^+} \frac{-x^2}{\tan(x)} = \boxed{\frac{-0}{0}}$$

$$\ln(y) = \lim_{x \to 0^+} \frac{\frac{d}{dx}[-x^2]}{\frac{d}{dx}[\tan(x)]}$$

$$\ln(y) = \lim_{x \to 0^+} \frac{-2x}{\sec^2(x)}$$

This ending is typical!!

$$\ln(y) = \frac{-2(0)}{\sec^2(0)}$$

$$\ln(y) = 0$$, Now, find y.

$$e^{\ln(y)} = e^0$$

$$y = 1$$

$$\boxed{\text{, So , } \lim_{x \to 0^+} [\sin(x)]^x = 1}$$

$\boxed{8.7}$

#71 *

$$\lim_{x\to\infty} x^{\frac{1}{x}} = \infty^{\frac{1}{\infty}} = \boxed{\infty^0}$$ stop!

let $y = \lim_{x\to\infty} x^{1/x}$

$$\ln(y) = \ln\left[\lim_{x\to\infty} x^{1/x}\right]$$

$$\ln(y) = \lim_{x\to\infty}\left\{\ln\left(x^{1/x}\right)\right\}$$

$$\ln(y) = \lim_{x\to\infty}\left[\frac{1}{x}\cdot\ln(x)\right]$$

$$\ln(y) = \lim_{x\to\infty} \frac{\ln(x)}{x} = \frac{\ln(\infty)}{\infty} = \boxed{\frac{\infty}{\infty}}$$ stop!

$$\ln(y) = \lim_{x\to\infty} \frac{\frac{d}{dx}[\ln(x)]}{\frac{d}{dx}(x)}$$

$$\ln(y) = \lim_{x\to\infty} \frac{1/x}{1}$$

$$\ln(y) = 0$$

$$e^{\ln(y)} = e^0$$

$$y = 1$$, $\boxed{\text{So, } \lim_{x\to\infty} x^{1/x} = 1}$

Have you seen this before? Test? Test? Test?