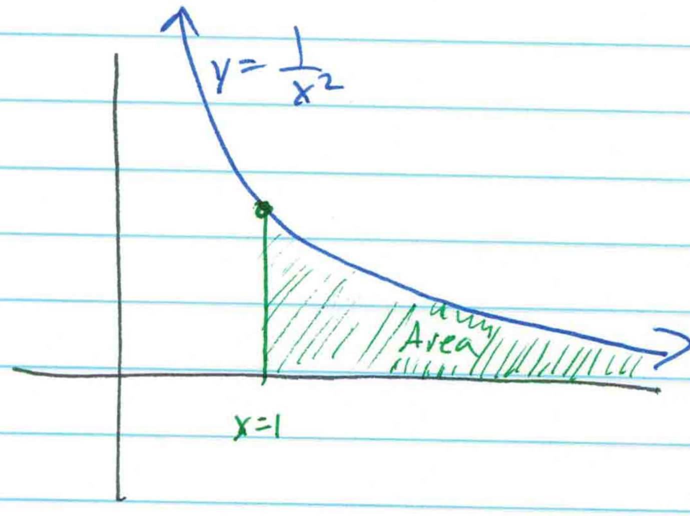
8.8 Improper Integrals

Example:

$$\int_1^\infty \frac{1}{x^2}\,dx$$



Problems with ∞?
Use limits??
watch This ?

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b x^{-2}\,dx$$

$$= \lim_{b\to\infty}\Big[-1\cdot x^{-1}\Big]_1^b$$

$$= \lim_{b\to\infty}\Big[\frac{-1}{x}\Big]_1^b$$

$$= \lim_{b\to\infty}\Big[\frac{-1}{(b)} - \Big(\frac{-1}{(1)}\Big)\Big]$$

$$= \lim_{b\to\infty}\Big(\frac{-1}{b} + 1\Big)$$

$$= 0 + 1$$

$$= 1$$

So, $\int_1^\infty \frac{1}{x^2}\,dx = 1$ ← The area under the curve is finite.

$\boxed{8.8}$

Problem with ∞, use limits??

#25

$$\int_4^{\infty} \frac{1}{x(\ln x)^3}\,dx = \lim_{b\to\infty} \int_4^{b} \frac{1}{x[\ln(x)]^3}\,dx$$

let $u=\ln(x)$	if $x=b$	if $x=4$
$\frac{du}{dx}=\frac{1}{x}$	$u=\ln(b)$	$u=\ln(4)$
$x\cdot du = dx$		

$$=\lim_{b\to\infty} \int_{u=\ln(4)}^{u=\ln(b)} \frac{1}{x\cdot u^3}(x\cdot du)$$

$$=\lim_{b\to\infty} \int_{\ln(4)}^{\ln(b)} u^{-3}\,du$$

$$=\lim_{b\to\infty} \left[\frac{u^{-2}}{-2}\right]_{\ln(4)}^{\ln(b)}$$

$$=-\frac{1}{2}\cdot\lim_{b\to\infty}\left[\frac{1}{u^2}\right]_{\ln(4)}^{\ln(b)}$$

$$=-\frac{1}{2}\cdot\lim_{b\to\infty}\left\{\frac{1}{[\ln(b)]^2}-\frac{1}{[\ln(4)]^2}\right\}$$

$$=-\frac{1}{2}\left[0-\frac{1}{[2\ln(2)]^2}\right]$$

$$=-\frac{1}{2}\left[-\frac{1}{4\ln^2(2)}\right] = \boxed{\frac{1}{8\ln^2(2)} \text{ or } \frac{1}{2\ln^2(4)}}$$

Area under the curve is finite?

$\boxed{8.8}$

#20

$$\int_0^{\infty} x e^{-\frac{x}{2}} dx = \lim_{b\to\infty} \int_0^b x e^{-\frac{x}{2}} dx$$

"Use Parts"

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Let $u = x$

$\frac{du}{dx} = 1$

$du = dx$

$dv = e^{-x/2} dx$

$\int dv = \int e^{-\frac{x}{2}} dx$

$v = \int e^{z} (-2dz)$

$v = -2\int e^z dz$

$v = -2e^z$

$v = -2e^{-x/2}$

let

$z = -\frac{x}{2}$

$\frac{dz}{dx} = -\frac{1}{2}$

$-2\,dz = dx$

$$= \lim_{b\to\infty} \left[(\overset{u}{x})(\overset{v}{-2e^{-x/2}}) \Big|_0^b - \int_0^b (\overset{v}{-2e^{-x/2}})(\overset{du}{dx}) \right]$$

$$= \lim_{b\to\infty} \left\{ \left[-2xe^{-x/2}\right]_0^b + 2\int_0^b e^{-\frac{x}{2}} dx \right\}$$

see above ↑

$$= \lim_{b\to\infty} \left\{ \left[-2xe^{-\frac{x}{2}}\right]_0^b + 2\left[-2e^{-\frac{x}{2}}\right]_0^b \right\}$$

$$= \lim_{b\to\infty} \left\{ \left[-2xe^{-\frac{x}{2}} - 4e^{-\frac{x}{2}}\right]_0^b \right\}$$

$$= \lim_{b\to\infty} \left\{ \left[-2e^{-x/2}(x+2)\right]_0^b \right\}$$

8.8 #20 cont'd

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$$= \lim_{b\to\infty} \left\{ -2e^{-b/2}(b+2) + 2e^{0/2}(0+2) \right\}$$

$$= \lim_{b\to\infty} \left\{ \frac{-2(b+2)}{e^{b/2}} + 2\cdot 1\cdot 2 \right\}$$

★★ $$= \lim_{b\to\infty} \frac{-2b-4}{e^{b/2}} + \lim_{b\to\infty} 4$$

$$= \lim_{b\to\infty} \frac{\frac{d}{db}(-2b-4)}{\frac{d}{db}(e^{b/2})} + 4$$

$$= \lim_{b\to\infty} \frac{-2}{\frac{1}{2}e^{b/2}} + 4$$

$$= 0 + 4$$

$$\boxed{= 4}$$

★★ stop!

$$\lim_{b\to\infty} \frac{-2b-4}{e^{b/2}} = \boxed{\frac{-\infty}{\infty}}$$

use L'Hopital's ~~Law~~ Rule!

Defintions:

I. $\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$, when f is cts on $[a,\infty)$

II. $\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$, when f is cts on $[-\infty, b]$

III. $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$, when f is cts on $(-\infty,\infty)$ and $c \in (-\infty,\infty)$

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$\boxed{8.8}$

- An improper integral converges if the limit exists.

- An improper integral diverges if the limit does not exist.

Definitions:

I. $\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx$, when f is cts on $[a,b)$ and f has an infinite discontinuity at b.

II. $\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx$, when f is cts on $(a,b]$ and f has an infinite discontinuity at a.

III. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, when f is cts on $[a,b]$ except for some $c \in [a,b]$ at which f has an infinite discontinuity.

Infinite Discontinuity:

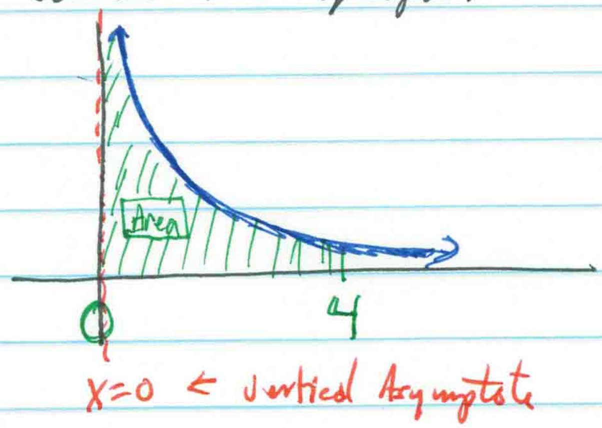
f has an infinite discontinuity at c if, from the right or left,

$\lim_{x \to c} f(x) = \infty$ or $\lim_{x \to c} f(x) = -\infty$

If f has a vertical Asymptote at $x = c$???

$\boxed{8.8}$

#5 $\displaystyle\int_0^4 \frac{1}{\sqrt{x}}\,dx$ → consider the graph:



$$= \lim_{c \to 0^+} \int_c^4 \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{c \to 0^+} \int_c^4 x^{-1/2}\,dx$$

$$= \lim_{c \to 0^+} \left[\frac{2}{1} \cdot x^{1/2} \right]_c^4$$

$$= \lim_{c \to 0^+} \left\{ 2\sqrt{(4)} - 2\sqrt{c} \right\}$$

$$= \lim_{c \to 0^+} \left(2 \cdot 2 - 2\sqrt{c} \right)$$

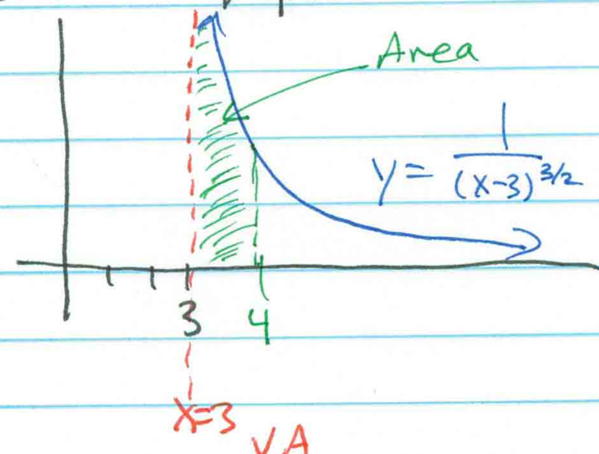
$$= 4 - 2\sqrt{0}$$

$$\boxed{= 4}$$ ← The area is finite.

The improper integral converges because the limits exists.

$\boxed{8.8}$

#6 $\displaystyle\int_3^4 \frac{1}{(x-3)^{3/2}}\,dx \longrightarrow$ consider the graph



$= \displaystyle\lim_{c\to 3^+} \int_c^4 (x-3)^{-3/2}\,dx$

let $u = x-3$ | if $x=4$ | if $x=c$
$\frac{du}{dx}=1$ | $u=4-3$ | $u=c-3$
$du=dx$ | $u=1$ |

$= \displaystyle\lim_{c\to 3^+} \int_{c-3}^{1} u^{-3/2}\,du$

$= \displaystyle\lim_{c\to 3^+} \left[-\frac{2}{1}\cdot u^{-1/2}\right]_{c-3}^{1}$

$= \displaystyle\lim_{c\to 3^+} \left[\frac{-2}{\sqrt{u}}\right]_{c-3}^{1}$

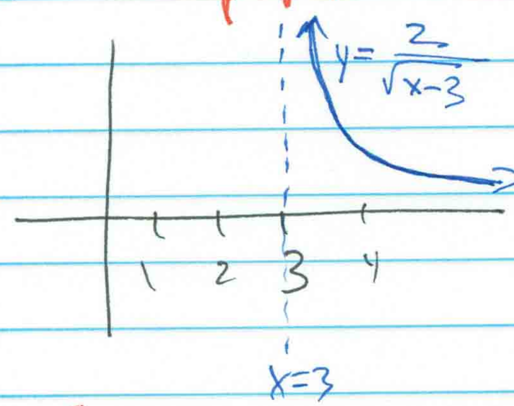
$= \displaystyle\lim_{c\to 3^+} \left(\frac{-2}{\sqrt{1}} + \frac{2}{\sqrt{c-3}}\right)$

☆☆ $= \displaystyle\lim_{c\to 3^+} \left(-2 + \frac{2}{\sqrt{c-3}}\right)$

$= -2 + \infty$

$= \infty$

☆☆ consider: $g(x)=\frac{2}{\sqrt{x-3}}$ & think of limits with a graph



So, $\displaystyle\lim_{x\to 3^+} \frac{2}{\sqrt{x-3}} = \infty$

← Area is infinite, the improper integral diverges.

$\boxed{8.8}$

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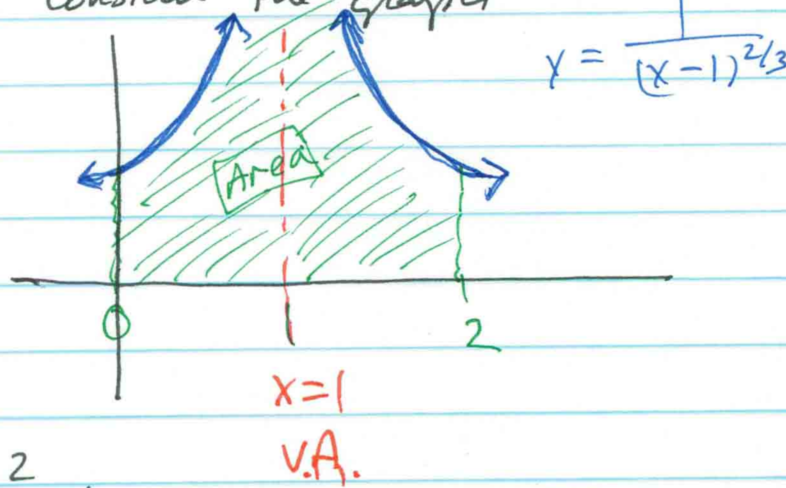
#8 $\int_0^2 \frac{1}{(x-1)^{2/3}}\,dx \longrightarrow$ consider the graph



$$= \int_0^1 \frac{1}{(x-1)^{2/3}}\,dx + \int_1^2 \frac{1}{(x-1)^{2/3}}\,dx$$

$$= \lim_{b\to 1^-} \int_0^b (x-1)^{-2/3}\,dx + \lim_{c\to 1^+} \int_c^2 (x-1)^{-2/3}\,dx$$

let $u = x-1$ $\frac{du}{dx} = 1$ $du = dx$	if $x=0$ $u=(0)-1$ $u=-1$	if $x=b$ $u=b-1$	if $x=c$ $u=c-1$	if $x=2$ $u=(2)-1$ $u=1$

$$= \lim_{b\to 1^-} \int_{-1}^{b-1} u^{-2/3}\,du + \lim_{c\to 1^+} \int_{c-1}^{1} u^{-2/3}\,du$$

$$= \lim_{b\to 1^-} \left[\frac{3}{1} u^{1/3}\right]_{-1}^{b-1} + \lim_{c\to 1^+} \left[3u^{1/3}\right]_{c-1}^{1}$$

$$= \lim_{b\to 1^-} \left[3\sqrt[3]{b-1} - 3\sqrt[3]{-1}\right] + \lim_{c\to 1^+} \left[3\sqrt[3]{1} - 3\sqrt[3]{c-1}\right]$$

$\boxed{8.8}$ 9/11

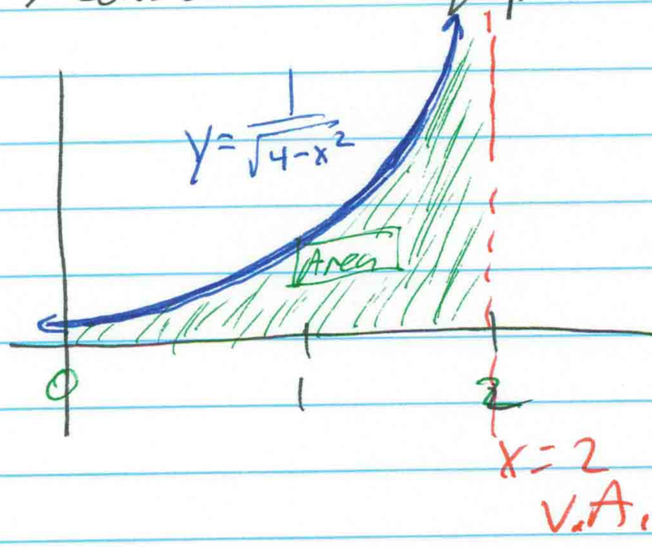
#8 cont'd

$= 3\sqrt[3]{1-1} - 3(-1) + 3(1) - 3\sqrt[3]{1-1}$

$= 3\cdot 0 + 3 + 3 - 3\cdot 0$

$\boxed{= 6}$ ← Area is finite.
The improper integral converges!

#42 $\displaystyle\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx$ → consider the graph



$\displaystyle = \lim_{b\to 2^-} \int_0^b \frac{1}{\sqrt{4-x^2}}\,dx$

$\boxed{\begin{array}{c} a^2 = 4 \\ a = 2 \end{array}}$

$\displaystyle = \lim_{b\to 2^-} \left[\arcsin\left(\frac{x}{2}\right)\right]_0^b$

$\displaystyle = \lim_{b\to 2^-} \left\{\arcsin\left(\frac{b}{2}\right) - \arcsin\left(\frac{0}{2}\right)\right\}$

$= \arcsin\left(\frac{2}{2}\right) - \arcsin(0)$

$= \arcsin(1) - \arcsin(0)$

$= \pi/2 - 0$

$\boxed{= \pi/2}$ ← Area is finite.
Improper Integral converges!

$\arcsin(0) = \theta$
$\sin(\arcsin(0)) = \sin(\theta)$
$0 = \sin(\theta)$
$\theta = 0$

$\arcsin(1) = \theta$
$\sin(\arcsin(1)) = \sin(\theta)$
$1 = \sin\theta$
$\theta = \pi/2$

$\boxed{8.8}$

#32 $\displaystyle\int_0^{\infty} \sin\left(\frac{x}{2}\right) dx = \lim_{b\to\infty} \int_0^{b} \sin\left(\frac{x}{2}\right) dx$

let $u = \frac{x}{2}$

$\frac{du}{dx} = \frac{1}{2}$

$2du = dx$

if $x = b$, $u = \frac{b}{2}$

if $x = 0$, $u = 0/2$, $u = 0$

$$= \lim_{b\to\infty} \int_0^{b/2} \sin(u) \cdot (2\,du)$$

$$= 2 \lim_{b\to\infty} \int_0^{b/2} \sin(u)\,du$$

$$= 2 \lim_{b\to\infty} \Big[-\cos(u)\Big]_0^{b/2}$$

$$= -2 \lim_{b\to\infty} \left[\cos\left(\frac{b}{2}\right) - \cos(0)\right]$$

$$= -2 \lim_{b\to\infty} \left[\cos\left(\frac{b}{2}\right) - 1\right]$$

= D.N.E. because the cosine function oscillates between -1 & 1 as $b\to\infty$.

Therefore, the Improper Integral Diverges.

#30 $\displaystyle\int_0^{\infty} \frac{e^x}{1+e^x} dx = \lim_{b\to\infty} \int_0^{b} \frac{e^x}{1+e^x} dx$

let $u = 1+e^x$

$\frac{du}{dx} = e^x$

$\frac{du}{e^x} = dx$

if $x = 0$, $u = 1+e^0$, $u = 1+1$, $u = 2$

if $x = b$, $u = 1+e^b$

$$= \lim_{b\to\infty} \int_2^{1+e^b} \frac{e^x}{u}\left(\frac{du}{e^x}\right)$$

#30 continued

$$= \lim_{b\to\infty} \int_2^{1+e^b} \frac{du}{u}$$

$$= \lim_{b\to\infty} \Big[\ln|u|\Big]_2^{1+e^b}$$

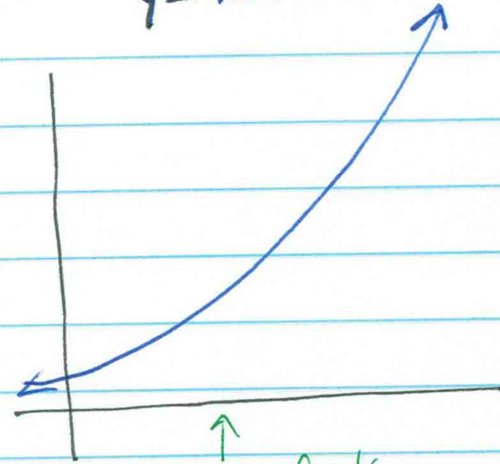
$$= \lim_{b\to\infty} \{\ln(1+e^b) - \ln(2)\}$$

$$= \infty - \ln 2$$

$$= \infty$$ ← Improper Integral Diverges.

The limit does not exist.

graph of $y = \ln(1+e^x)$



The function increases without bound as $x \to \infty$.

Cool Theorem for later use:

$$\int_1^\infty \frac{dx}{x^p} = \begin{cases} \dfrac{1}{p-1}, & \text{if } p > 1, \text{ Integral converges} \\ \text{diverges}, & \text{if } p \le 1 \end{cases}$$