$\boxed{9.1}$ Sequences

- A **sequence** is a function whose domain is the set of positive integers.
  It will usually be denoted with subscript notation rather than function notation.
  For example:

$$
\begin{array}{lcl}
f(1) = a_1 & \text{or} & a(1) = a_1 \\
f(2) = a_2 & \text{or} & a(2) = a_2 \\
f(3) = a_3 & \text{or} & a(3) = a_3 \\
\vdots & & \\
f(n) = a_n & \text{or} & a(n) = a_n
\end{array}
$$

- $a_1, a_2, a_3 \ldots a_n$ are terms, and <u>a_n is the n-th term.</u>

- An entire sequence can be denoted as $\{a_n\}$

Examples:

- $\{a_n\} = \{1 - \frac{1}{n}\}$ are
  $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\ldots$

- $\{a_n\} = \{(-1)^n n\}$ are

  $0, -1, 2, -3, 4, \ldots\ldots$

$\boxed{9.1}$

- Some sequences are recursively defined

$\{d_n\}_{15}^{\infty}$ defined as $d_1 = 25$
and $d_{n+1} = d_n - 5$

So, $d_2 = d_1 - 5$ | $d_3 = d_2 - 5$ | $d_4 = d_3 - 5$
$d_2 = 25 - 5$ | $d_3 = 20 - 5$ | $d_4 = 15 - 5$
$d_2 = 20$ | $d_3 = 15$ | $d_4 = 10$

etc ....

- For the majority of this chapter we'll be looking at sequences that have limiting values. These sequences are said to converge.

Example:

$\{a_n\} = \{\frac{1}{2^n}\}$ are

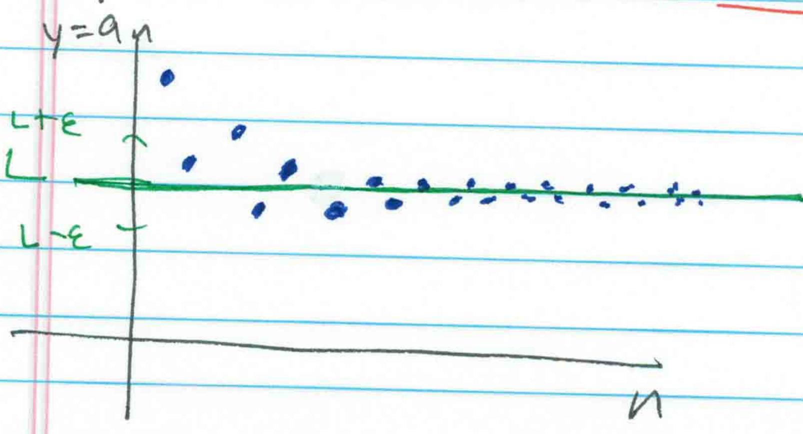
$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$

This sequence converges to 0.

Definition: Let L be a real number. The limit of a sequence $\{a_n\}$ is L, written as
$\lim_{n \to \infty} a_n = L$ if and only if
for each $\epsilon > 0$, there exists $M > 0$
such that $|a_n - L| < \epsilon$ whenever $n > M$.

If the limit L of a sequence exists, then the sequence converges to L.
If the limit of a sequence does not exist, then the sequence diverges.

- If we plot the terms of a convergent sequence we will see a "Horizontal Asymptote."



- Example:

$$\{a_n\} = \left\{\frac{n+4}{n+1}\right\}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+4}{n+1}$$

$$= 1$$

← check plot in "sequence mode" on TI-83

Theorem 9.1

- Let L be a real number. Let f be a function of a real variable such that $\lim_{x\to\infty} f(x) = L$.

If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer n, then $\lim_{n\to\infty} a_n = L$.

In other words, if a sequence $\{a_n\}$ agrees with a function f at every positive integer, and if $f(x) \to L$ as $x \to \infty$, then $\{a_n\} \to L$ as well.

$\boxed{9.1}$

Consider $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$

we know $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$ from earlier.

let $y = \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$

$\ln(y) = \ln\left[\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x\right]$

$\ln(y) = \lim_{x\to\infty} \left[\ln\left(1+\frac{1}{x}\right)^x\right]$

$\ln(y) = \lim_{x\to\infty} \left[x \cdot \ln\left(1+\frac{1}{x}\right)\right]$

$\ln(y) = \lim_{x\to\infty} \left[\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}\right]$ use L'Hopital's Rule $\boxed{\frac{0}{0}}$

$\ln(y) = \lim_{x\to\infty} \left\{\frac{\frac{d}{dx}\left[\ln\left(1+\frac{1}{x}\right)\right]}{\frac{d}{dx}\left[\frac{1}{x}\right]}\right\}$

$\ln(y) = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}} \cdot \cancel{(-1x^{-2})}}{\cancel{(-1x^{-2})}}$

$\ln(y) = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}}$

$\ln(y) = 1$

$e^{\ln(y)} = e^1$

$y = e$, so $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$

AND

$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$, by <u>Theorem 9.1</u>.

$\boxed{9.1}$

properties of limits of sequences

Let $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = K$

(i) $\lim\limits_{n\to\infty} (a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n = L \pm K$

(ii) $\lim\limits_{n\to\infty} (c \cdot a_n) = c \cdot \lim\limits_{n\to\infty} a_n = c \cdot L$

where c is any real number.

(iii) $\lim\limits_{n\to\infty} (a_n \cdot b_n) = \left(\lim\limits_{n\to\infty} a_n\right) \cdot \left(\lim\limits_{n\to\infty} b_n\right) = L \cdot K$

(iv) $\lim\limits_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n} = \frac{L}{K}$

where $b_n \neq 0$, and $K \neq 0$.

New Notation - FACTORIAL

$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1) \cdot n$

$0! = 1$

$3! = 1 \cdot 2 \cdot 3 = 6$

$5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$

$2n! = 2(n!) = 2 \cdot [1 \cdot 2 \cdot 3 \cdot \cdots (n-1) \cdot n]$

$(2n)! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1) \cdot n\,(n+1) \cdots (2n-1) \cdot 2n$

$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$

$6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$

$2! = 1 \cdot 2 = 2$

$1! = 1 = 1$

9.1

Squeeze Theorem for Sequences

If $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$ and

there exists an integer N such that

$$a_n \le c_n \le b_n \text{ for all } n > N,$$

then

$$\lim_{n\to\infty} c_n = L.$$

Example: Consider $\lim_{n\to\infty} \frac{\sin(n)}{n}$

$\lim_{n\to\infty} \frac{1}{n} = 0$, $\lim_{n\to\infty} \frac{-1}{n} = 0$

Since $-\frac{1}{n} \le \frac{\sin(n)}{n} \le \frac{1}{n}$ for all $n \ge 1$

then

$$\lim_{n\to\infty} \frac{-1}{n} \le \lim_{n\to\infty} \frac{\sin(n)}{n} \le \lim_{n\to\infty} \frac{1}{n}$$

$$0 \le \lim_{n\to\infty} \frac{\sin(n)}{n} \le 0$$

and $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$

by the Squeeze Theorem.

9.1

Absolute Value Theorem

For the sequence $\{a_n\}$, if

$$\lim_{n\to\infty} |a_n| = 0, \text{ then } \lim_{n\to\infty} a_n = 0.$$

Find:

#40. $\displaystyle\lim_{n\to\infty} \frac{5n}{\sqrt{n^2+4}}$

Consider for all $n \geq 1$

$$n^2+4n+4 \geq n^2+4 \geq n^2$$

$$(n+2)^2 \geq n^2+4 \geq n^2$$

$$\sqrt{(n+2)^2} \geq \sqrt{n^2+4} \geq \sqrt{n^2}$$

$$n+2 \geq \sqrt{n^2+4} \geq n$$

$$\frac{5n}{n+2} \leq \frac{5n}{\sqrt{n^2+4}} \leq \frac{5n}{n}$$

Division!

$$\begin{array}{r} 5 \\ n+2 \overline{\smash{)}\, 5n + 0} \\ -5n - 10 \\ \hline -10 \end{array}$$

So

$$\boxed{\frac{5n}{n+2} = 5 \frac{-10}{n+2}}$$

★★ $$\boxed{5 - \frac{10}{n+2} \leq \frac{5n}{\sqrt{n^2+4}} \leq 5}$$ ★★ Use the squeeze Theorem

$$\lim_{n\to\infty}\left(5 - \frac{10}{n+2}\right) = 5 \ \& \ \lim_{n\to\infty} 5 = 5$$

So, by the Squeeze Theorem,

$$\lim_{n\to\infty} \frac{5n}{\sqrt{n^2+4}} = 5$$

$\boxed{9.1}$

#60 Find $\lim\limits_{n\to\infty} \frac{(n-2)!}{n!}$

$$\lim_{n\to\infty} \frac{(n-2)!}{n!} = \lim_{n\to\infty} \frac{\cancel{1\cdot 2\cdot 3}\cdots\cancel{(n-4)}\cancel{(n-3)}\cancel{(n-2)}}{\cancel{1\cdot 2\cdot 3}\cdots\cancel{(n-4)}\cancel{(n-3)}\cancel{(n-2)}(n-1)n}$$

$$= \lim_{n\to\infty} \frac{1}{n(n-1)}$$

$$= \lim_{n\to\infty} \frac{1}{n^2-n}$$

$$\lim_{n\to\infty} \frac{(n-2)!}{n!} = 0$$

← Check a plot in "Sequence mode"

#62 Find $\lim\limits_{n\to\infty} \frac{n^2}{2n+1} - \frac{n^2}{2n-1}$

$$\lim_{n\to\infty} \left[\frac{n^2}{2n+1} - \frac{n^2}{2n-1}\right] = \lim_{n\to\infty} \left[\frac{n^2(2n-1) - n^2(2n+1)}{(2n+1)\cdot(2n-1)}\right]$$

$$= \lim_{n\to\infty} \frac{2n^3 - n^2 - 2n^3 - n^2}{4n^2-1}$$

$$= \lim_{n\to\infty} \left[\frac{-2n^2}{4n^2-1}\right] \cdot \left[\frac{\frac{1}{n^2}}{1/n^2}\right]$$

$$= \lim_{n\to\infty} \frac{-2}{4 - 1/n^2}$$

$$= \frac{-2}{4-0}$$

$$= -\frac{1}{2}$$

$\boxed{9.1}$

Example: Find $\lim_{n\to\infty} \cos(\pi n)$

$$\{\cos(\pi n)\} = \{-1, 1, -1, 1, -1, \ldots\}$$ check with a plot on Ti-83

So, $\lim_{n\to\infty} \cos(\pi n)$ does not exist

and $\{\cos(\pi n)\}$ diverges.

#52. Find $\lim_{n\to\infty} \dfrac{1\cdot 3\cdot 5 \cdots (2n-1)}{n!}$

consider

$$\frac{1\cdot 3\cdot 5\cdot 7\cdots(2n-1)}{n!} = \frac{1\cdot 3\cdot 5\cdot 7\cdots(2n-1)}{1\cdot 2\cdot 3\cdot 4\cdots(n)}$$

$\geq 3/2 \quad \geq 3/2 \quad \geq 3/2 \quad \geq 3/2$

$n-1$ factors

is $\dfrac{2n-1}{n} \geq \dfrac{3}{2}$???

$$2\cdot(2n-1) \geq 3\cdot n$$
$$4n-2 \geq 3n$$
$$n \geq 2$$

, So for $n \geq 2$, $\dfrac{2n-1}{n} \geq \dfrac{3}{2}$.

Therefore

$$\frac{1\cdot 3\cdot 5\cdot 7\cdots(2n-1)}{1\cdot 2\cdot 3\cdot 4\cdots n} \geq \left(\frac{3}{2}\right)^{n-1}$$

Since $\lim_{n\to\infty} \left(\frac{3}{2}\right)^{n-1} = \infty$,

then $\lim_{n\to\infty} \dfrac{1\cdot 3\cdot 5\cdots(2n-1)}{n!} = \infty$.

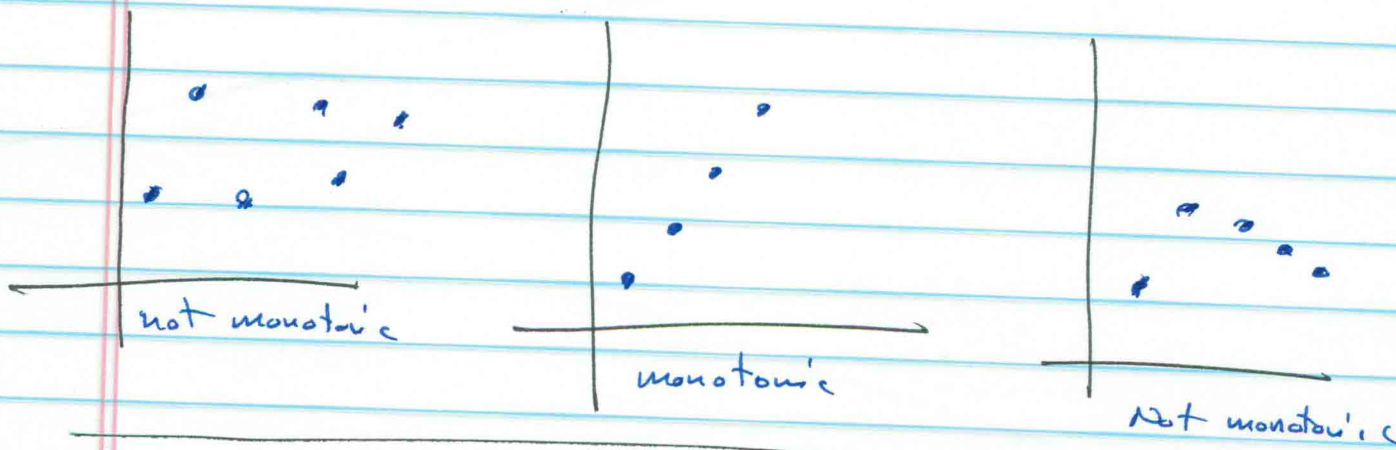
9.1

Definition: A sequence is monotonic if its terms are non-decreasing,

$$a_1 \le a_2 \le a_3 \le a_4 \le \dots \le a_n \le \dots$$

or if its terms are non-increasing

$$a_1 \ge a_2 \ge a_3 \ge a_4 \ge \dots \ge a_n \ge \dots$$



Definition:

(i) A sequence $\{a_n\}$ is bounded above if there is a real number M such that $a_n \le M$ for all n. The number M is called an upper bound.

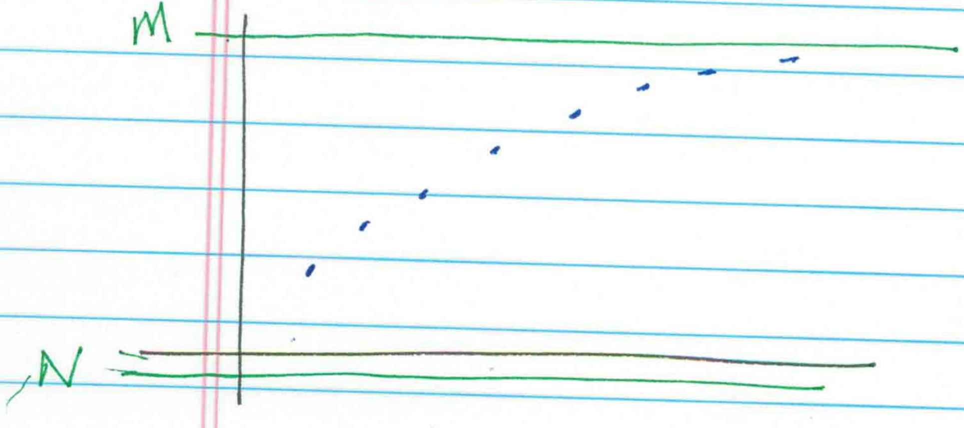
(ii) A sequence $\{a_n\}$ is bounded below if there is a real number N such that $N \le a_n$ for all n. The number N is called a Lower bound.

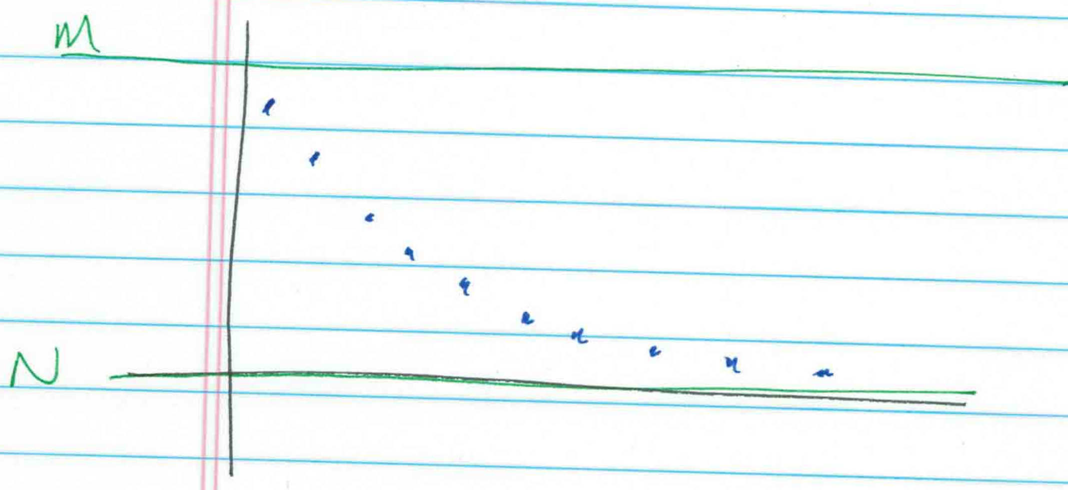
(iii) A sequence $\{a_n\}$ bounded if it is bounded above and below.

9.1

Theorem 9.5

If a sequence $\{a_n\}$ is bounded and monotonic, then it converges.





$\boxed{9.1}$

#86

$a_n = ne^{-\frac{n}{2}}$, consider $f(x) = xe^{-\frac{x}{2}}$

$$f'(x) = (x)\cdot\frac{d}{dx}\left(e^{-\frac{x}{2}}\right) + \left(e^{-\frac{x}{2}}\right)\frac{d}{dx}(x)$$

$$f'(x) = x\left(e^{-\frac{x}{2}}\right)\cdot\frac{d}{dx}\left(-\frac{x}{2}\right) + e^{-\frac{x}{2}}\cdot 1$$

$$f'(x) = x\left(e^{-x/2}\right)\cdot\left(-\frac{1}{2}\right) + e^{-\frac{x}{2}}$$

$$\boxed{f'(x) = e^{-x/2}\cdot\left(-\frac{x}{2}+1\right)}$$ ← Tells us about increasing & decreasing

$$0 = e^{-\frac{x}{2}}\left(-\frac{x}{2}+1\right)$$

Either $e^{-x/2} = 0$, or $-\frac{x}{2}+1 = 0$

false

$$1 = \frac{x}{2}$$

$$\boxed{2 = x}$$

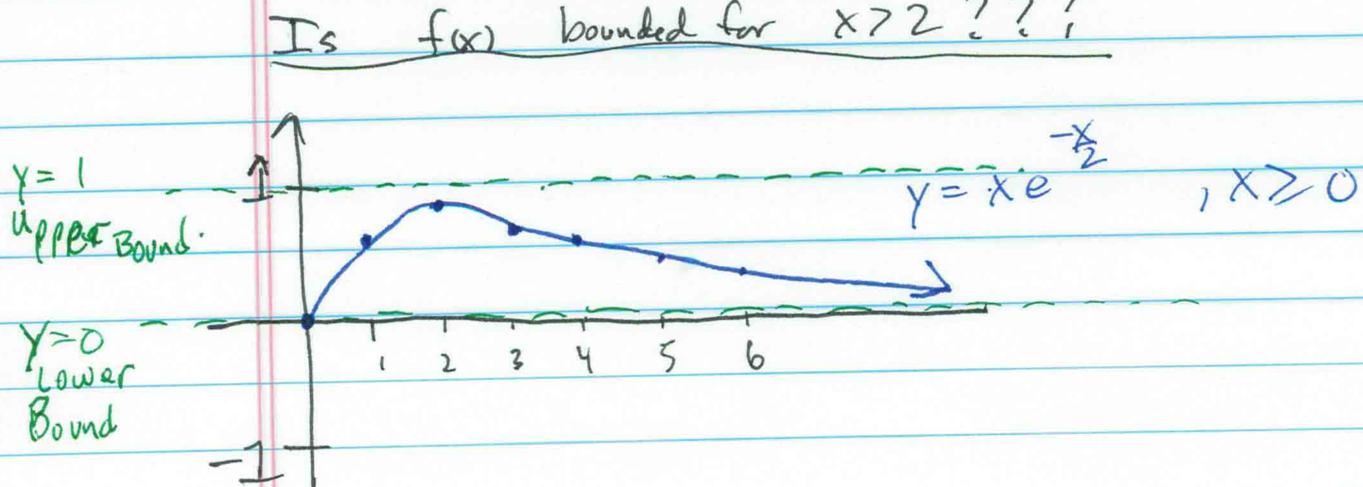
Test $x=3$: $f'(3) = e^{-(3)/2}\cdot\left(-\frac{(3)}{2}+1\right)$

$$f'(3) = \frac{1}{e^{3/2}}\cdot\left(-\frac{1}{2}\right)$$

$f'(3) < 0$, and we can see $f(x)$ is decreasing for $x > 2$.

This means that $f(x)$ is monotonic for $x > 2$.

Is $f(x)$ bounded for $x > 2$???



$\boxed{9.1}$

#86 cont'd

- from the graph of $y = xe^{-\frac{x}{2}}$, for $x \geq 0$ we can see that the function is bounded above by $y=1$ and bounded below by $y=0$.

Therefore, by Theorem 9.5, $\{ne^{-n/2}\}$ is a convergent sequence. $\{ne^{-n/2}\}$ is bounded and monotonic for $n \geq 2$.

- Fibonacci Sequence - #121

Consider $a_{n+2} = a_{n+1} + a_n$, with $a_1 = 1$ & $a_2 = 1$

$\{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \text{-----}\}$ is the Fibonacci Sequence.

- Consider the following ratio of consecutive terms:

let $\frac{a_{n+1}}{a_n} = b_n$

$$\frac{a_{n+2}}{a_{n+1}} = \frac{a_{n+1}}{a_{n+1}} + \frac{a_n}{a_{n+1}}$$

$$\frac{a_{n+2}}{a_{n+1}} = 1 + \frac{1}{\left(\frac{a_{n+1}}{a_n}\right)}$$

9.1 #121 continued

$$b_{n+1} = 1 + \frac{1}{b_n}$$

– Suppose that $\lim_{n\to\infty} b_n = r$ exists ***, $r \neq 0$.

then we can also see that $\lim_{n\to\infty} b_{n+1} = \lim_{b\to\infty} b_n$

and $\lim_{n\to\infty} b_{n+1} = r$.

Therefore $\lim_{n\to\infty} b_{n+1} = \lim_{n\to\infty}\left(1+\frac{1}{b_n}\right)$

$$r = 1 + \frac{1}{r}$$ ⇐ Now, solve for r.

$$r^2 = r + 1$$

$$r^2 - r - 1 = 0$$

$$r = \frac{1 \pm \sqrt{5}}{2} \rightarrow \boxed{r = \frac{1+\sqrt{5}}{2}}$$

$\frac{1+\sqrt{5}}{2}$ is the "Golden Ratio."

*** $\lim_{n\to\infty} b_n = r$ ⇐ you can try to justify ~~the~~ existence of this limit