$\boxed{9.10}$ Taylor and MacLaurin Series

In $\boxed{9.9}$ we found power series representations of functions by "manipulating" the geometric power series representation of $f(x) = \frac{1}{1-x}$ using $\sum_{n=0}^{\infty} ar^n$.

Can we find power series representations for any functions? Which functions? How can we do this?

Theorem 9.22: The Form of a Convergent Power Series

If f is represented by a power series $f(x) = \sum a_n(x-c)^n$ for all x in a open interval I containing c, then

$$a_n = \frac{f^{(n)}(c)}{n!}$$ and

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \dots$$

Definitions of Taylor & Maclaurin Series:

If a function f has derivatives of all orders at $x=c$, then the series

$$\sum_{n=0}^{\infty} \frac{f^{n}(c)}{n!}(x-c)^n$$ is called the Taylor series for $f(x)$ at c. If $c=0$, the series is called the Maclaurin series for $f(x)$.

Example: Find a Maclaurin Series for $f(x) = \sin x$ and determine the interval of convergence.

9.10

$c = 0$

$f(x) = \sin x \qquad f(0) = \sin 0 = 0$

$f'(x) = \cos x \qquad f'(0) = \cos 0 = 1$

$f''(x) = -\sin x \qquad f''(0) = -\sin 0 = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -\cos 0 = -1$

$f^{IV}(x) = \sin x \qquad f^{IV}(0) = \sin 0 = 0$

$\vdots$

So $\sin(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{IV}(0)}{4!}(x-0)^4 + \cdots$

$\sin(x) = 0 + x + \frac{0}{2}(x^2) + \frac{(-1)}{3!}(x)^3 + \frac{0}{4!}(x^4) + \cdots$

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots$

$$\boxed{\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}}$$

Find the interval of convergence. $u_n = \frac{x^{2n+1}}{(2n+1)!}$

consider: $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left| \frac{x^{2(n+1)+1}}{[2(n+1)+1]!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right|$

$= \lim_{n\to\infty} \left| \frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right|$

$= \lim_{n\to\infty} \left| \frac{x^2 \quad (2n+1)!}{(2n+3)(2n+2)\cdot(2n+1)!} \right|$

$= \lim_{n\to\infty} |x^2| \cdot \left(\frac{1}{[2n+3][2n+2]} \right)$

$= |x^2| \cdot \lim_{n\to\infty} \frac{1}{(2n+3)(2n+2)}$

$= |x^2| \cdot 0 \quad = \underline{\underline{0}} \rightarrow$ Interval of convergence is $(-\infty, \infty)$.

9.10

Question: If $f(x)$ is a function that has derivatives of all orders, when is it true that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(c)}{n!}(x-c)^n \quad ?$$

We know from Theorem 9.19, for each n,

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{n}(c)}{n!}(x-c)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$$

← we know z depends on both x & n.

If $R_n \to 0$, then we'll know that the Taylor series converges to $f(x)$ for all $x \in I$, the interval of convergence.

Theorem 9.23: Convergence of Taylor Series

If $\lim_{n\to\infty} R_n = 0$ for all x in the interval I, then the Taylor series for f converges to and equals $f(x)$.

So,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(c)}{n!}(x-c)^n$$

Back to Example 1: Show that the Maclaurin series for $f(x) = \sin(x)$ converges to $\sin x$ for all x.

We'll show

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

is true for all x.

9.10

That is, we can show $R_n(x) = \frac{f^{n+1}(z)}{(n+1)!}(x-c)^{n+1} \to 0$

we have $R_n(x) = \frac{f^{n+1}(z)}{(n+1)!} x^{n+1}$

Since $f^{n+1}(x) = \pm\sin(x)$ or $f^{n+1}(x) = \pm\cos(x)$

we know that $|f^{n+1}(z)| \le 1$ for any real number z.

Therefore, for any fixed x, we can use Theorem 9.19

$$|R_n(x)| = \left| \frac{f^{n+1}(z)\, x^{n+1}}{(n+1)!} \right| \le \left| \frac{1 \cdot x^{n+1}}{(n+1)!} \right| = \frac{|x^{n+1}|}{(n+1)!}$$

So, for all x, $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$.

By Theorem 9.23, we can see that the Maclaurin series for $\sin x$ converges to $\sin x$ for all x.

Main Series	Interval of Convergence
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	$(-\infty, \infty)$
$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$(-\infty, \infty)$
$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$(-\infty, \infty)$
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$	$(-1, 1)$
$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$	$(-1, 1)$
$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$[-1, 1]$

more on Pg. 682

$\boxed{9.10}$ we can make use of substitution, differentiation, and integration to create series that represent many other functions.

Example: If $f(x) = \sin(x^2)$, find its MacLaurin Series.

We know

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}$$

$$\sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}.$$

#22 $g(x) = e^{-3x}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-3x} = \sum_{n=0}^{\infty} \frac{(-3x)^n}{n!}$$

$$e^{-3x} = \sum_{n=0}^{\infty} (-1)^n \frac{3^n x^n}{n!}$$

#32 $h(x) = x\cos(x)$

$$x\cos(x) = x \cdot \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!}$$

$0! = 1$
$1! = 1$
$2! = 1 \cdot 2 = 2$
$3! = 1 \cdot 2 \cdot 3 = 6$
$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$
$5! = 24 \cdot 5 = 120$

$\boxed{9.10}$

#38 $g(x) = e^x \cos x$

$$g(x) = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n!}\right)$$

$$g(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$$

$$\Rightarrow 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

$$1 \cdot \quad - \frac{x^2}{2} \quad + \frac{x^4}{24} \quad - \frac{x^6}{720} + \cdots$$

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots$$

$$- \frac{x^2}{2} \quad - \frac{x^3}{2} \quad - \frac{x^4}{4} \quad - \frac{x^5}{12} - \cdots$$

$$+ \quad + \frac{x^4}{24} + \frac{x^5}{24} + \cdots$$

$$g(x) = 1 + x + \left(\frac{x^3}{6} - \frac{3x^3}{6}\right) + \left(\frac{2x^4}{24} - \frac{6x^4}{24}\right) + \left(\frac{x^5}{120} - \frac{10x^5}{120} + \frac{5x^5}{24\cancel{5}}\right) + \cdots$$

$$g(x) = 1 + x + \left(\frac{-2x^3}{6}\right) + \left(\frac{-4x^4}{24}\right) + \left(\frac{-4x^5}{120}\right)$$

$$g(x) = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \frac{x^5}{30} + \cdots$$

#56 $\displaystyle\int_0^{1/2} \frac{\arctan(x)}{x}\,dx$ ← Use a power series to approximate the value of this integral with an error less than 0.0001.

Use $\displaystyle\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

$\boxed{9.10}$

$$\frac{1}{x}\cdot \arctan(x) = \frac{1}{x}\cdot \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\frac{\arctan(x)}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n+1}$$

So

$$\int_0^{1/2} \frac{\arctan(x)}{x}\,dx = \int_0^{1/2} \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n+1}\right) dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)\cdot(2n+1)} \Bigg|_0^{1/2}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)^2} \Bigg|_0^{1/2}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{2}\right)^{2n+1}}{(2n+1)^2} - \sum_{n=0}^{\infty} (-1)^n \frac{(0)^{2n+1}}{(2n+1)^2} \to 0$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)^2\, 2^{2n+1}}$$

The "error" is less than or equal to "the first omitted term"

$$|R_N| \le a_{N+1}$$

$$a_{N+1} < 0.0001$$

$$\frac{1}{[2(N+1)+1]^2\, 2^{2(N+1)+1}} < \frac{1}{10,000}$$

$$\frac{1}{(2N+3)^2\, 2^{2N+3}} < \frac{1}{10,000}$$

Solve for n. $\quad 10,000 < (2N+3)^2 2^{2N+3}$

If $\quad n = 3 \quad$ we have

$$10,000 < 9^2 \cdot 2^9$$

$$10,000 < 41,472$$

$\boxed{9.10}$

So,

$$\int_0^{1/2} \frac{\arctan(x)}{x}\,dx \approx \sum_{n=0}^{3} (-1)^n \frac{1}{(2n+1)^2 2^{n+1}}$$

$$\approx \frac{1}{2} - \frac{1}{3^2 2^3} + \frac{1}{5^2 2^5} - \frac{1}{7^2 2^7}$$

$$\approx 0.\underline{4872}01672336$$

From the calculator, we get

$$\int_0^{1/2} \frac{\arctan(x)}{x}\,dx \approx 0.\underline{4872}22358295$$

So, we really have our "error" within $\underline{\underline{0.0001}}$.