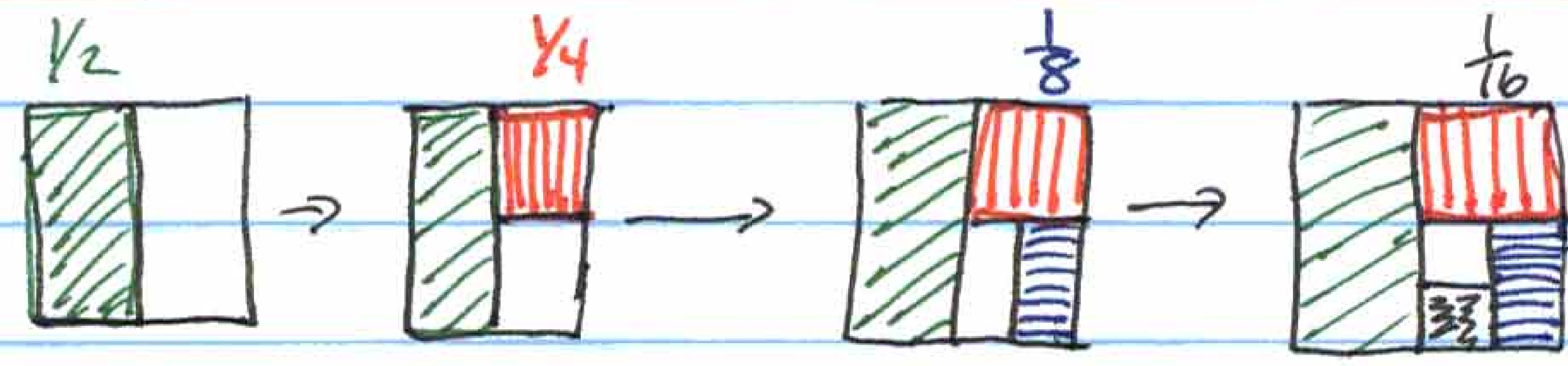
$\boxed{9.2}$ Series and Convergence

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots + \cdots$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$



If we keep on adding half of what we added previously, what will happen?

Definition: Let $\{a_n\}$ be an infinite sequence.

For $n \in \mathbb{N}$,

$$S_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

is the n-th partial sum of the infinite series $\sum_{i=1}^{\infty} a_i$.

The series $\sum_{i=1}^{\infty} a_i$ converges if $\lim_{n\to\infty} S_n = S$ exists.

$\{S_n\}$ is the sequence of partial sums.

$\lim_{n\to\infty} S_n = S$ means that $\{S_n\}$ converges to S.

S is called the sum of the series.

If $\{S_n\}$ diverges, then the series $\sum_{i=1}^{\infty} a_i$ diverges.

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- Geometric Sequences and Series -

- A sequence $\{b_1, b_2, b_3, \ldots\ldots\}$ is called a geometric sequence if for every $n \in \mathbb{N}$,

$$\frac{b_{n+1}}{b_n} = r \quad \text{where } r \text{ is a constant.}$$

This constant r is called the ratio of the geometric sequence.

Example: $3, 6, 12, 24, 48, \ldots\ldots$

$$\frac{6}{3} = 2, \quad \frac{12}{6} = 2, \quad \frac{24}{12} = 2, \quad \frac{48}{24} = 2$$

So, $r = 2$

Theorem: If $\{b_n \mid n \in \mathbb{N}\}$ is a geometric sequence, then

$$b_n = b_1 \cdot r^{n-1}$$, where r is the ratio of the sequence.

Example: $b_1 = 3$, $r = 2$

$b_4 = b_1 r^{4-1}$

$b_4 = b_1 r^3$

$b_4 = (3) \cdot (2)^3$

$b_4 = 3 \cdot 8$

$b_4 = 24$

9.2 Consider, $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots + ar^n + \cdots$, $a \neq 0$

$\sum_{n=0}^{\infty} ar^n$ is a geometric series with ratio r

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Fact: $\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}$, $r \neq 1$

Proof:

let $S_n = \sum_{i=0}^{n-1} r^i = \underbrace{\overset{i=0}{1} + \overset{i=1}{r} + \overset{i=2}{r^2} + \cdots + \overset{i=n-2}{r^{n-2}} + \overset{i=n-1}{r^{n-1}}}_{n \text{ terms}}$

$$r \cdot S_n = r(1 + r + r^2 + \cdots + r^{n-2} + r^{n-1})$$
$$= r + r^2 + r^3 + \cdots + r^{n-1} + r^n$$

Consider $S_n - rS_n = S_n(1-r)$

$$(1 + \cancel{r} + \cancel{r^2} + \cdots + \cancel{r^{n-2}} + \cancel{r^{n-1}}) - (\cancel{r} + \cancel{r^2} + r^3 + \cdots + \cancel{r^{n-1}} + r^n) = S_n(1-r)$$
$$1 - r^n = S_n(1-r)$$

So, $\boxed{S_n = \frac{1-r^n}{1-r}}$

Theorem 9.6: A geometric series with ratio $|r| \geq 1$ diverges. If $0 < |r| < 1$, then the series

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

converges to the sum.

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Proof: Let $S_n = \sum_{i=0}^{n-1} ar^i$

$$= a + ar + ar^2 + \dots + ar^{n-2} + ar^{n-1}$$

$$= a(1 + r + r^2 + \dots + r^{n-2} + r^{n-1})$$

$$S_n = a\left(\frac{1-r^n}{1-r}\right)$$

Case 1: If $|r|>1$, then $r^n \to \infty$ as $n \to \infty$

So, $\lim_{n\to\infty} S_n = \infty$

Case 2: If $|r|=1$, then

$$S_n = \underbrace{a+a+a+a+a+a+a \dots +a+a}_{n \text{ term}},$$

subcase 1, $S_n = n \cdot a$, or

subcase 2, $S_n = a-a+a-a+a-a+a \dots$

subcase 1: $\lim_{n\to\infty} S_n = \lim_{n\to\infty} n\,a = \infty$

subcase 2: $\lim_{n\to\infty} S_n$ does not exist due to oscillation

Case 3: If $0<|r|<1$,

then $\lim_{n\to\infty} r^n = 0$, and

we have $\lim_{n\to\infty} S_n = \lim_{n\to\infty} a\left(\frac{1-r^n}{1-r}\right)$

$$= \boxed{\frac{a}{1-r}}$$

we will use the result through the rest of the chapter.

$\boxed{9.2}$ #40

$$\sum_{n=0}^{\infty} 6\left(\frac{4}{5}\right)^n = \frac{a}{1-r} \quad , \quad a = 6 \, , \; r = \frac{4}{5}$$

$$= \frac{6}{1-\left(\frac{4}{5}\right)}$$

Check on Ti-83

$$= \frac{6}{\frac{5}{5}-\frac{4}{5}}$$

$$= \frac{6}{\frac{1}{5}}\left(\frac{\frac{5}{1}}{\frac{5}{1}}\right)$$

$$= 30$$

#40***

$$\sum_{n=0}^{\infty} 6\left(-\frac{4}{5}\right)^n = \frac{a}{1-r} \quad , \quad a=6 \, , \; r = -\frac{4}{5}$$

Check on Ti-83

$$= \frac{6}{1-\left(-\frac{4}{5}\right)}$$

$$= \frac{6}{\frac{5}{5}+\frac{4}{5}}$$

$$= \frac{6}{\frac{9}{5}}\left(\frac{\frac{5}{1}}{\frac{5}{1}}\right)$$

$$= \frac{30}{9}$$

$$= \frac{10}{3}$$

#52 $0.\overline{9} = 0.999999\ldots$

$$= 0.9 + 0.09 + 0.009 + 0.0009 + \cdots$$

$$= \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10,000}$$

$$\Rightarrow a = \frac{9}{10} \quad , \quad r = \frac{9/100}{9/10} = \left(\frac{\frac{9}{100}}{\frac{9}{10}}\right)\left(\frac{\frac{100}{9}}{\frac{100}{9}}\right) = \frac{1}{10}$$

$\boxed{9.2}$ #52 continued

$$0.\overline{9} = \sum_{n=0}^{\infty} \frac{9}{10}\left[\frac{1}{10}\right]^n$$

$$= \frac{a}{1-r}$$

$$= \frac{\left(\frac{9}{10}\right)}{1-\left(\frac{1}{10}\right)}$$

$$= \frac{\frac{9}{10}}{\frac{10}{10}-\frac{1}{10}}$$

$$= \frac{\frac{9}{10}}{9/10}$$

$$\boxed{0.\overline{9} = 1}$$

#55 $0.0\overline{75} = 0.075 + 0.00075 + 0.0000075 + \cdots$

$$= \frac{75}{1000} + \frac{75}{100,000} + \frac{75}{10,000,000} + \cdots$$

$$a = 0.075 \quad , \quad r = \frac{\frac{75}{100,000}}{\frac{75}{1000}} = \left(\frac{\frac{75}{100,000}}{\frac{75}{1000}}\right)\left(\frac{\frac{100,000}{75}}{\frac{100,000}{75}}\right) = \frac{1}{100}$$

$$0.0\overline{75} = \sum_{n=0}^{\infty} \frac{75}{1000}\left(\frac{1}{100}\right)^n$$

$$= \frac{a}{1-r}$$

$$= \frac{\frac{75}{1000}}{1-\left(\frac{1}{100}\right)}$$

$$= \frac{\frac{75}{1,000}}{\frac{100}{100}-\frac{1}{100}}$$

$$= \left(\frac{\frac{75}{1000}}{\frac{99}{100}}\right)\left(\frac{\frac{1000}{1}}{\frac{1000}{1}}\right)$$

$$= \frac{75}{990} = \frac{\cancel{3}\cdot\cancel{5}\cdot 5}{2\cdot 3\cdot\cancel{3}\cdot\cancel{5}\cdot 11} = \frac{5}{66}$$

$\boxed{9.2}$

#82 (a) Find all values of x for which

$$\sum_{n=0}^{\infty} 4\left(\frac{x-3}{4}\right)^n \quad \text{converges.}$$

(b) For these values of x, write the sum of the series.

(a) $\sum_{n=0}^{\infty} 4\cdot\left(\frac{x-3}{4}\right)^n = \sum_{n=0}^{\infty} ar^n$ when

$$\boxed{a=4 \ \& \ \frac{x-3}{4}=r}$$

This series converges when $|r|<1$.

solve for x:

$$\left|\frac{x-3}{4}\right|<1$$

$$4\cdot\left|\frac{x-3}{4}\right|<4\cdot 1$$

$$|x-3|<4$$

$$-4<x-3<4$$

$$-4+3<3+x-3<3+4$$

$$-1< x<7$$

(b) If $x \in (-1,7)$, then

$$\sum_{n=0}^{\infty} 4\left(\frac{x-3}{4}\right)^n = \frac{a}{1-r}$$

$$= \left[\frac{(4)}{1-\left(\frac{x-3}{4}\right)}\right]\cdot\left(\frac{\frac{4}{1}}{\frac{4}{1}}\right)$$

$$= \frac{16}{4-(x-3)}$$

$$= \frac{16}{4-x+3}$$

$$\sum_{n=0}^{\infty} 4\left(\frac{x-3}{4}\right)^n = \frac{16}{7-x}$$

9.2

Theorem 9.7 - Properties of Infinite Series

If $\sum a_n = A$, $\sum b_n = B$, and c is a real number, then the following series converge to the indicated sums.

① $$\sum_{n=1}^{\infty} ca_n = c \cdot \sum_{n=1}^{\infty} a_n = cA$$

② $$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = A + B$$

③ $$\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n = A - B$$

$\boxed{\text{Telescoping Series}}$ "Sometimes, you get Lucky."

Example form: $(b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + \cdots + (\quad)$

cancel each other out

$$\boxed{S = b_1 - \lim_{n \to \infty} b_{n+1}}$$

#24 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$, re-write using partial fractions

$$\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$$

$$1 = A(n+2) + Bn$$

, let $n=0$, let $n=-2$

$n=0$	$n=-2$
$1 = A(0+2) + B \cdot 0$	$1 = A(-2+2) + B(-2)$
$1 = 2A$	$1 = -2B$
$\frac{1}{2} = A$	$-\frac{1}{2} = B$

$\boxed{9.2}$

~~#24~~ cont'd

So, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \sum_{n=1}^{\infty} \left(\frac{\frac{1}{2}}{n} - \frac{\frac{1}{2}}{n+2} \right)$

$$\boxed{= \sum_{n=1}^{\infty} \left[\frac{1}{2n} - \frac{1}{2(n+2)} \right]}$$ use for b_{n+1}

$= \left(\frac{1}{2} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{8}\right) + \left(\frac{1}{6} - \frac{1}{10}\right) + \left(\frac{1}{8} - \frac{1}{12}\right) + \left(\frac{1}{10} - \frac{1}{14}\right) + \cdots\cdots$

Don't cancel out , telescoping form

So, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \lim_{n\to\infty} S_n$ The first terms that didn't cancel out

$= \displaystyle\lim_{n\to\infty} \left[b_1 - b_{n+1} \right]$

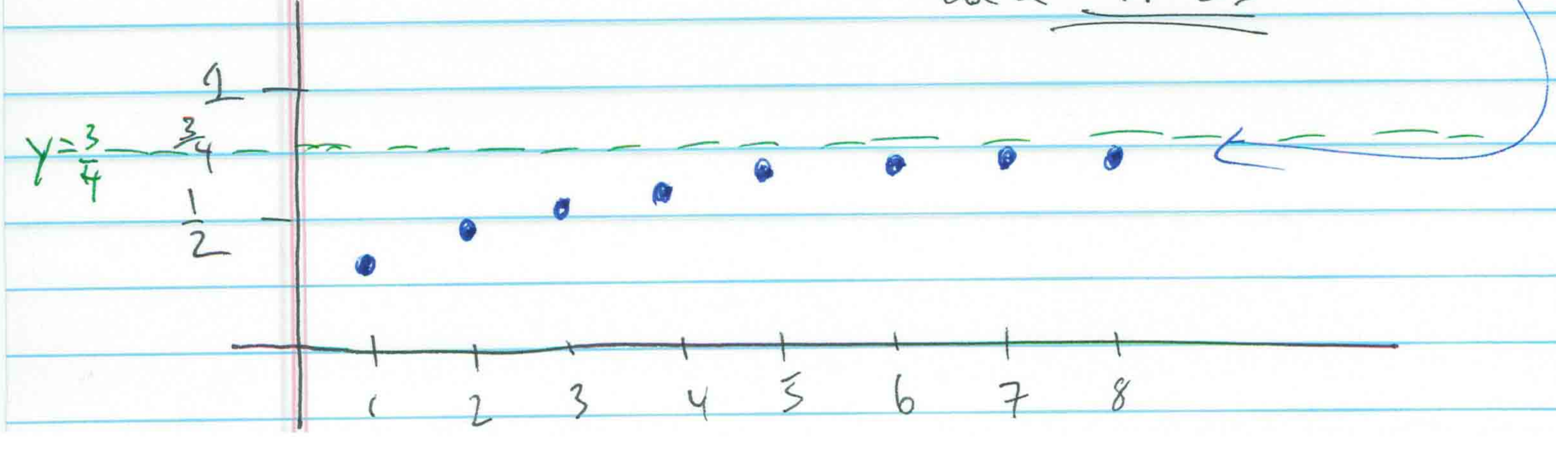
$$\boxed{\begin{aligned} b_{n+1} &= \frac{1}{2(n+1)} - \frac{1}{2[(n+1)+2]} \\ &= \frac{1}{2n+2} - \frac{1}{2n+4} \end{aligned}}$$

$= \displaystyle\lim_{n\to\infty} \left[\frac{1}{2} + \frac{1}{4} - \frac{1}{2n+2} - \frac{1}{2n+4} \right]$

$= \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

Check this with a graph of the sequence of partial sums.

use a TI-83



$\boxed{9.2}$

n-th Term Test for Divergence

← very important, will be used frequently!

1st Series convergence implies the sequence's n-th term tends to zero.

$$\boxed{\text{Theorem 9.8: If } \sum_{n=1}^{\infty} a_n \text{ converges, then } \lim_{n\to\infty} a_n = 0.}$$

Theorem 9.9: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges

Here's the test.

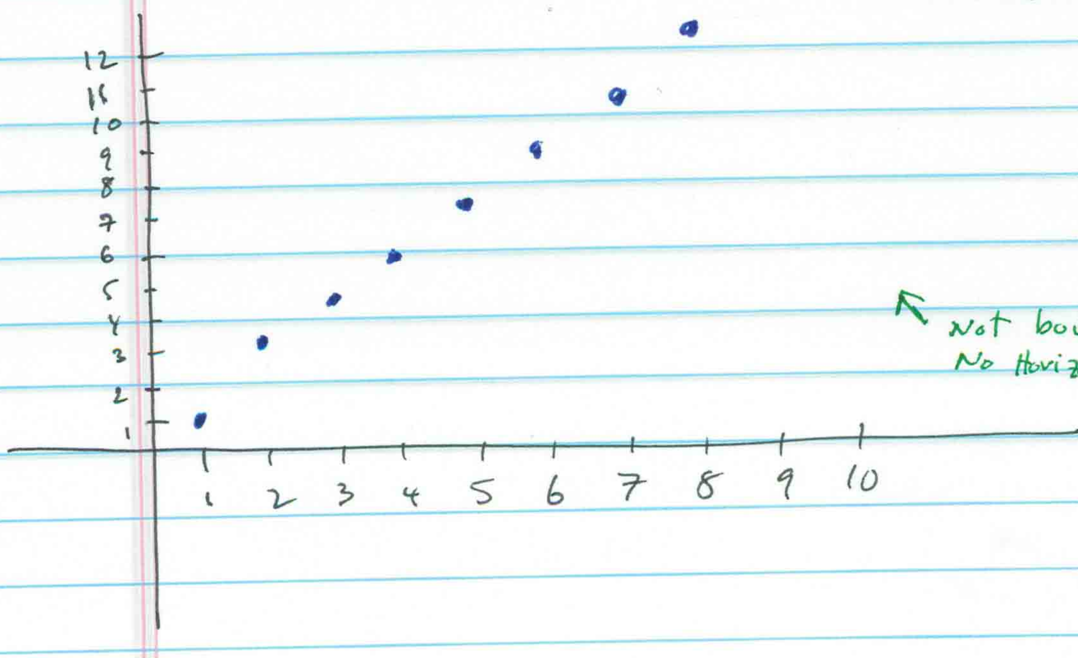
#65 $\sum_{n=0}^{\infty} (1.075)^n$ is a geometric series

$$\sum_{n=0}^{\infty} (1.075)^n = \sum_{n=0}^{\infty} ar^n$$

$a=1$, $r=1.075$

since $|r| \geq 1$,

the series diverges.



Sequence of Partial Sums

Not bounded, No Horizontal Asymptote

$\boxed{9.2}$

#66

$$\sum_{n=1}^{\infty} \frac{2^n}{100} = \frac{1}{100} \cdot \sum_{n=1}^{\infty} 2^n$$

$$= \frac{1}{100} \cdot \left(2^1 + 2^2 + 2^3 + 2^4 + \cdots \right)$$

↑ these get very big quickly

$$a_n = \frac{2^n}{100}$$

$\lim_{n\to\infty} \frac{2^n}{100} \neq 0$, So by the n-th term test, we can see that this series diverges.

#58

$$\sum_{n=1}^{\infty} \frac{n+1}{2n-1}$$

1st: consider the n-th term test.

Let $a_n = \frac{n+1}{2n-1}$

$$\lim_{n\to\infty} \frac{n+1}{2n-1} = \lim_{n\to\infty} \left(\frac{n+1}{2n-1}\right)\left(\frac{\frac{1}{n}}{\frac{1}{n}}\right)$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2-\frac{1}{n}}$$

$$= \frac{1}{2}$$

Since $\lim_{n\to\infty} a_n \neq 0$, the series diverges.