9.4 Comparison of Series

Theorem 9.12: Direct Comparison Test:

Let $0 < a_n \le b_n$, for all n

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges

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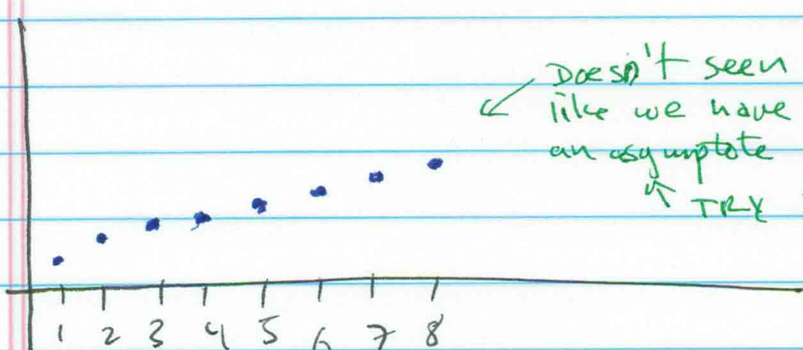
1. If the "larger" series converges, then the "smaller" series converges.
2. If the "smaller" series diverges, then the "larger" series diverges.

#12 Consider $\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n}-1}$

think about graph of partial sums sequence.

~~what does it tell us?~~



Doesn't seem like we have an asymptote

TRY to prove divergence, ??

will show "diverges"

So, $b_n = \frac{1}{4\sqrt[3]{n}-1}$, do we know $a_n = ?$

$4\sqrt[3]{n} \ge 4\sqrt[3]{n} - 1$

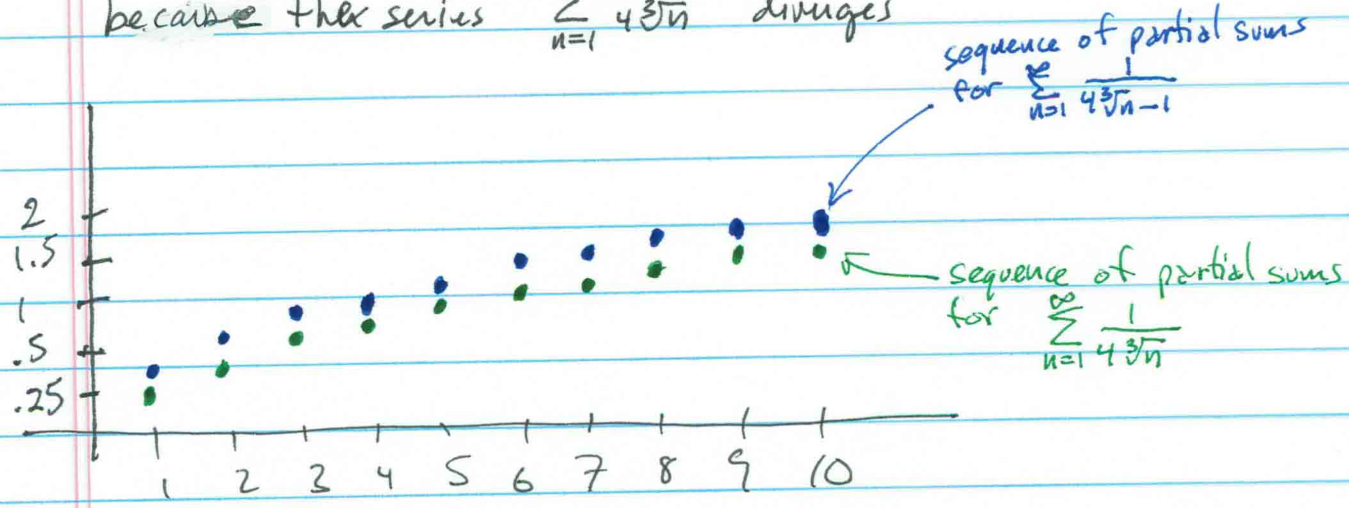
$\frac{1}{4\sqrt[3]{n}} \le \frac{1}{4\sqrt[3]{n}-1}$

will lead to a type of p-series

9.4 #12 cont'd

Here's our p-series $\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n}} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$
where $p = \frac{1}{3}$
by Theorem 9.11 for p-series $\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n}}$ is divergent.

So, by the Direct Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n}-1}$ diverges because the series $\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n}}$ diverges



#10 Consider $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$ ← Does this remind you of a p-series? Convergent? Divergent?

So, $b_n = \frac{1}{\sqrt{n^3}}$ ← $p = 3/2$

$\sqrt{n^3} = n^{3/2}$

$n^3 \le n^3+1$

$\sqrt{n^3} \le \sqrt{n^3+1}$

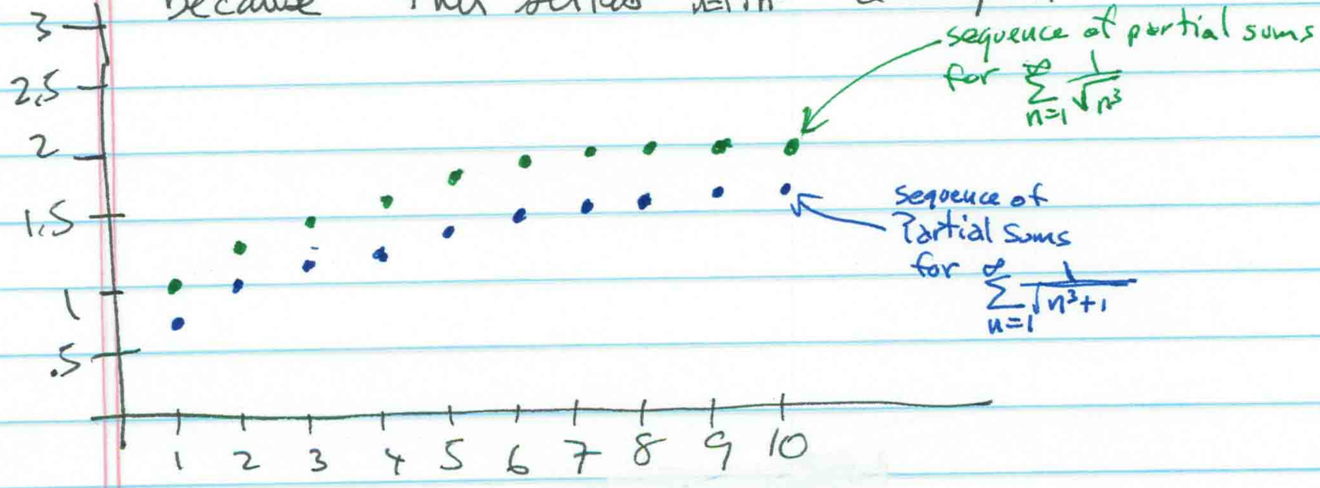
$\frac{1}{\sqrt{n^3}} \ge \frac{1}{\sqrt{n^3+1}}$

So $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}} \ge \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$

By Theorem 9.11, the p-series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$ is a convergent series.

$\boxed{9.4}$ #10 cont'd

So, by the Direct Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$ converges because the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$ converges.



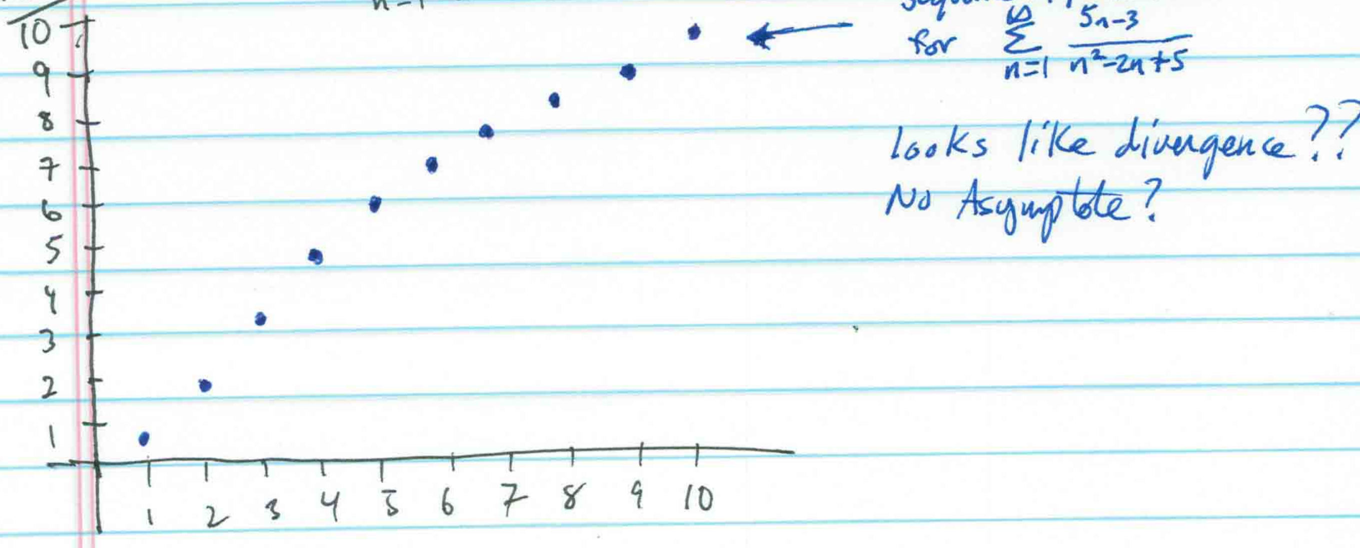
Theorem 9.13: Limit Comparison Test

Suppose that $a_n > 0$, $b_n > 0$, and

$$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = L$$

where L is $\boxed{\text{finite and positive}}$. Then, the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

#20 Consider $\sum_{n=1}^{\infty} \frac{5n-3}{n^2-2n+5}$

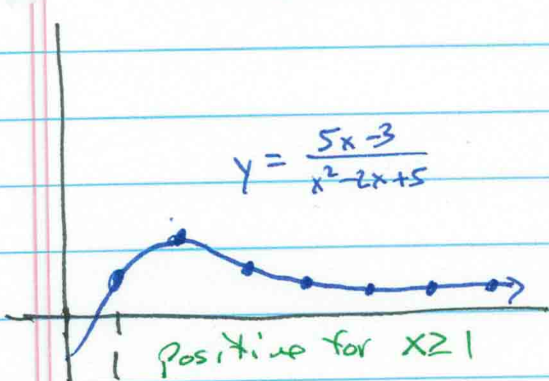


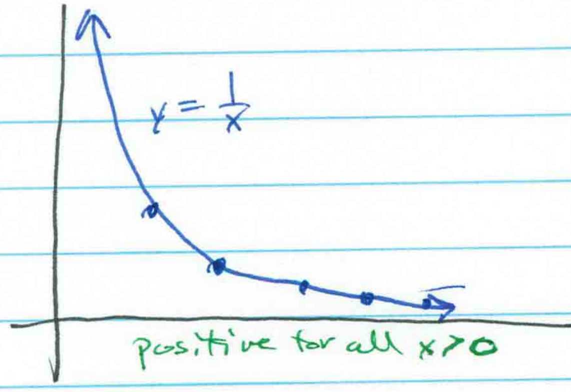
$\boxed{9.4}$ #20 cont'd

Try to show divergence by "comparing" with a divergent series. Try using $b_n = \frac{1}{n}$, because $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent harmonic series.

① Show $a_n \geq 0$ and $b_n \geq 0$.

$$\frac{5n-3}{n^2-2n+5} \geq 0 \quad \& \quad \frac{1}{n} \geq 0$$





② Evaluate the limit.

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{\frac{5n-3}{n^2-2n+5}}{\frac{1}{n}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{5n-3}{n^2-2n+5}\right)\cdot\left(\frac{n}{1}\right)$$

$$= \lim_{n\to\infty} \frac{5n^2-3n}{n^2-2n+5}$$

$$= \lim_{n\to\infty}\left(\frac{5n^2-3n}{n^2-2n+5}\right)\left(\frac{\frac{1}{n^2}}{\frac{1}{n^2}}\right)$$

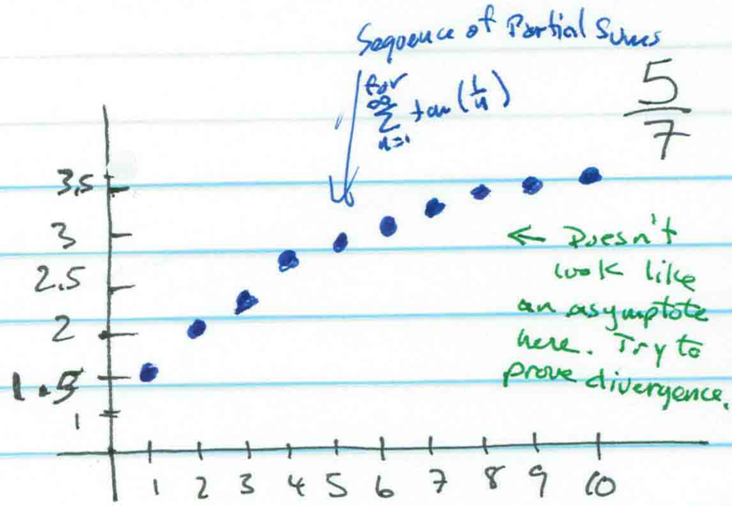
$$= \lim_{n\to\infty} \frac{5-3/n}{1-2/n+5/n^2}$$

$$= 5 \quad \longleftarrow$$ the limit is finite & positive!

So, by Theorem 9.13 the Limit Comparison Test, both series Diverge.

$\boxed{9.4}$ #28

Consider $\sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$



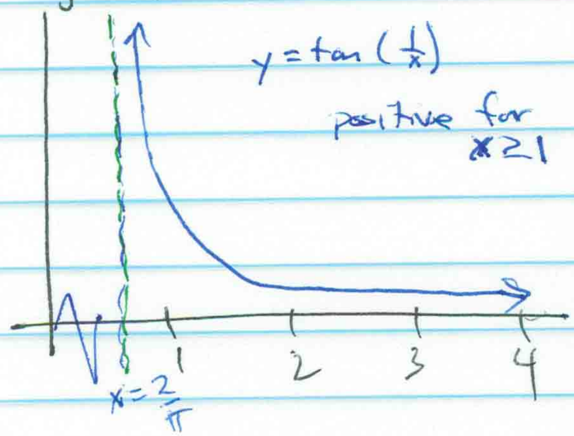
Try to show divergence by comparing with a divergent series. Try using $\sum_{n=1}^{\infty} \frac{1}{n}$ again.

① Show $a_n \geq 0$ & $b_n \geq 0$.

$\tan\left(\frac{1}{n}\right) \geq 0$ ⟷ for $n \geq 1$

&

$\frac{1}{n} \geq 0$, for $n \geq 1$



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② Evaluate the limit:

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left[\frac{\tan\left(\frac{1}{n}\right)}{\frac{1}{n}}\right]$$

← Indeterminate form $\frac{0}{0}$, use L'Hopital's Rule

$$= \lim_{n\to\infty} \frac{\frac{d}{dn}\left[\tan\left(\frac{1}{n}\right)\right]}{\frac{d}{dn}\left[\frac{1}{n}\right]}$$

cancels

$$= \lim_{n\to\infty} \frac{\sec^2\left(\frac{1}{n}\right) \cdot \cancel{\frac{d}{dn}\left[\frac{1}{n}\right]}}{\cancel{\frac{d}{dn}\left[\frac{1}{n}\right]}}$$

$$= \lim_{n\to\infty} \sec^2\left(\frac{1}{n}\right)$$

$$= \sec^2\left[\lim_{n\to\infty} \frac{1}{n}\right]$$

$$= \sec^2(0)$$

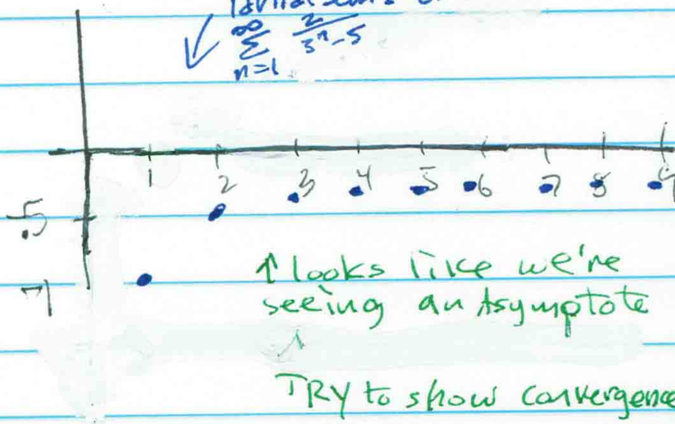
$$= 1$$ ← the limit is finite & Positive!

So, by Theorem 9.13, the Limit Comparison Test, both series Diverge.

$\boxed{9.4}$ #16

Consider $\sum_{n=1}^{\infty} \frac{2}{3^n - 5}$

sequence of partial sums for $\sum_{n=1}^{\infty} \frac{2}{3^n-5}$



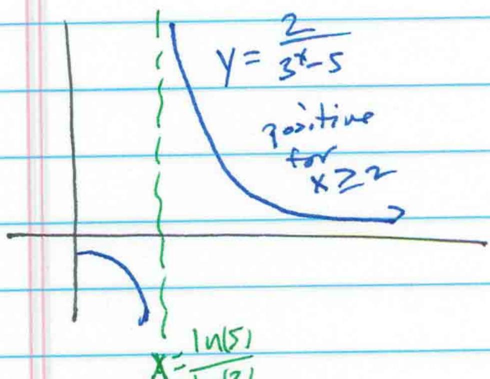
↑ looks like we're seeing an asymptote

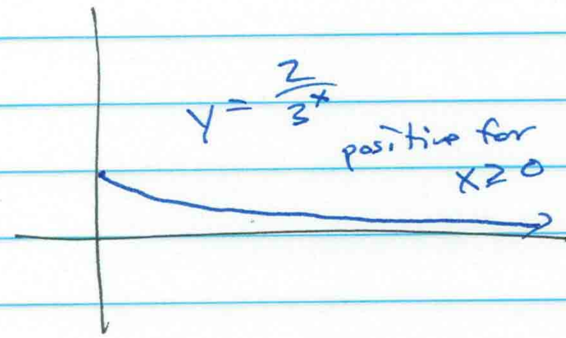
TRY to show convergence.

Consider $\sum_{n=1}^{\infty} \frac{2}{3^n} = 2 \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$

where $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ is a convergent geometric series with $r = \frac{1}{3}$

① Show $a_n \geq 0$ & $b_n \geq 0$





② Evaluate the limit.

$$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty} \left(\frac{\frac{2}{3^n-5}}{\frac{2}{3^n}}\right)$$

$$= \lim_{n\to\infty} \left(\frac{2}{3^n-5}\right)\left(\frac{3^n}{2}\right)$$

$$= \lim_{n\to\infty} \left(\frac{3^n}{3^n-5}\right)\left(\frac{\frac{1}{3^n}}{1/3^n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{1 - 5/3^n}$$

$$= 1$$

The limit is finite & positive!

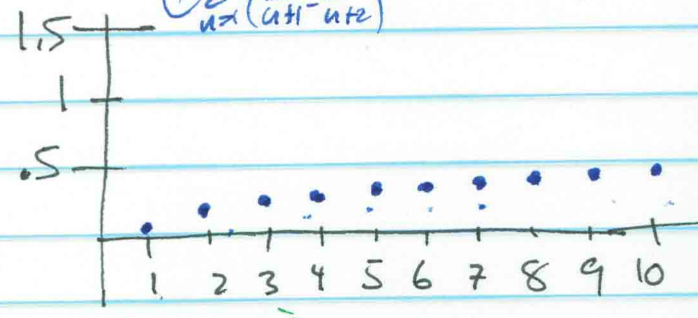
So by Theorem 9.13, the Limit Comparison Test, both series converge.

$\boxed{9.4}$ #34

Consider $\sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$

looks like a telescoping series ???

Sequence of partial sums for $\sum_{n=1}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$



looks like we're seein an Asymptote — Try to show convergence.

let

Partial sum

$$S_N=\sum_{n=1}^{N}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$$

$$=\underbrace{\left(\frac{1}{2}-\frac{1}{3}\right)}_{n=1}+\overbrace{\left(\frac{1}{3}-\frac{1}{4}\right)}^{n=2}+\overbrace{\left(\frac{1}{4}-\frac{1}{5}\right)}^{n=3}+\dots+\overbrace{\left(\frac{1}{N}-\frac{1}{N+1}\right)}^{n=N-1}+\overbrace{\left(\frac{1}{N+1}-\frac{1}{N+2}\right)}^{n=N}$$

So,

$$S_N=\frac{1}{2}-\frac{1}{N+2}$$

Now, evaluate the limit of the partial sum

$$S=\lim_{N\to\infty}S_N=\lim_{N\to\infty}\left(\frac{1}{2}-\frac{1}{N+2}\right)$$

$$S=\frac{1}{2}$$

This means that $\sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$ is a convergent, telescoping series. Its sum is $\frac{1}{2}$.

TRY #36 JUST FOR FUN