9.5

Alternating Series Test

Theorem 9.14 Alternating Series Test

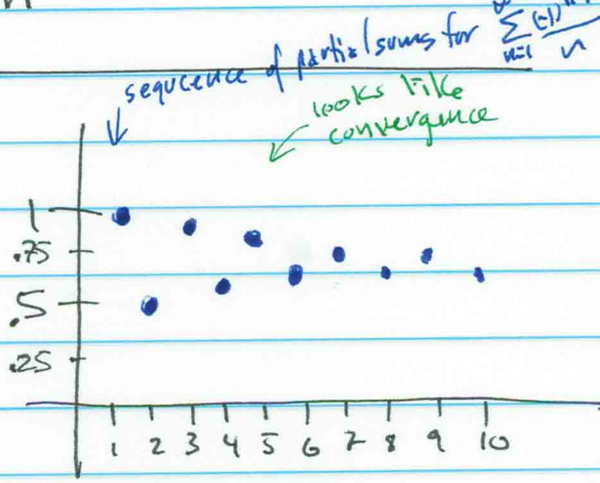
Let $a_n > 0$. The alternating series

$$\sum_{n=1}^{\infty} (-1)^n a_n \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converge if the following two condition are met

① $\lim_{n\to\infty} a_n = 0$, AND

② $a_{n+1} \le a_n$, for all n



Example: Consider $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

Try to ~~satisfy~~ both criteria for Theorem 9.14:

① $\lim_{n\to\infty} a_n = 0$

Evaluate: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n}$

$$= 0$$

② Show $a_{n+1} \le a_n$, for all n

$$n+1 \ge n$$

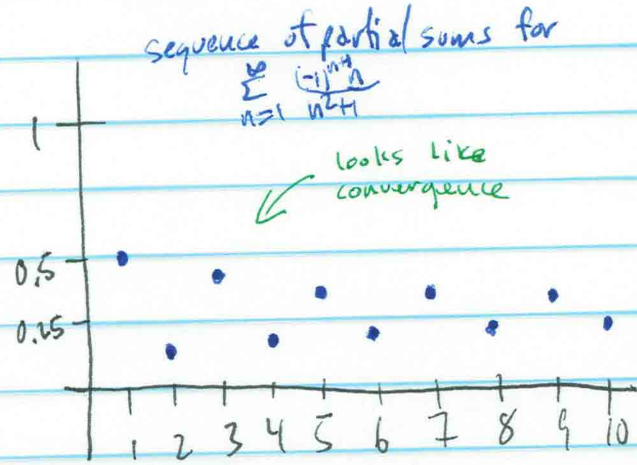
$$\frac{1}{n+1} \le \frac{1}{n}$$

So, $a_{n+1} \le a_n$!!

Therefore, by Theorem 9.14, the Alternating Series Test, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

$\boxed{9.5}$

#16 consider $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2+1}$



TRY TO SATISFY both criteria for Theorem 9.14

① $\lim_{n\to\infty} a_n = 0$

Evaluate: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n^2+1}$

$= \lim_{n\to\infty} \left(\frac{n}{n^2+1}\right)\cdot\left(\frac{\frac{1}{n^2}}{\frac{1}{n^2}}\right)$

$= \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n^2}}$

$= \frac{0}{1}$

$= 0$ ✓

② Show $a_{n+1} \le a_n$, for all $n \ge 1$

show: $\frac{n+1}{(n+1)^2+1} \le \frac{n}{n^2+1}$

[You could also use differentiation to show decreasing.]

$n^3 + 2n^2 + n + n \ge n^3 + n^2 + n + 1$, for all $n \ge 1$

$n^3 + 2n^2 + 2n \ge n^3 + n^2 + n + 1$

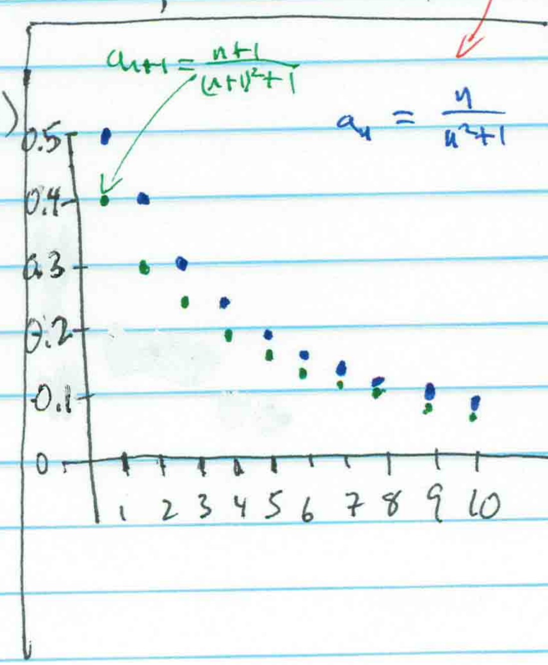
$n(n^2+2n+2) \ge (n^2+1)(n+1)$

$\frac{n}{n^2+1} \ge \frac{n+1}{n^2+2n+2}$

$\frac{n}{n^2+1} \ge \frac{n+1}{(n^2+2n+1)+1}$

$\frac{n}{n^2+1} \ge \frac{n+1}{(n+1)^2+1}$

$a_n \ge a_{n+1}$



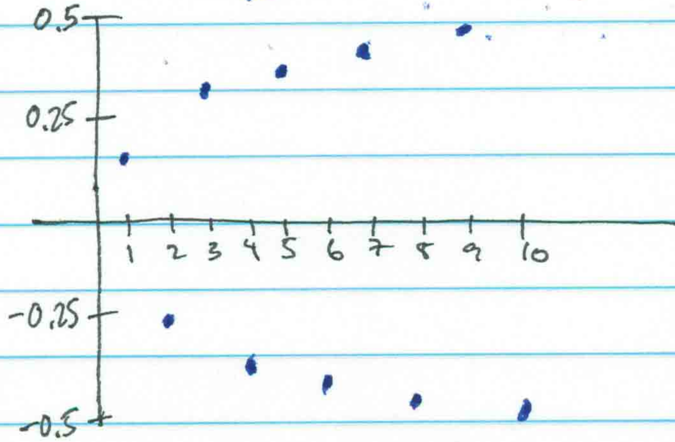
$\boxed{9.5}$ #16 cont'd

Therefore, by Theorem 9.14, the Alternating Series Test, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2+1}$ converges.

#18 Consider $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^2+5}$.

sequence of partial sums for $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^2+5}$

looks like divergence??



Use N^{th}-Term Test for Divergence??

- If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges. If $\lim_{n\to\infty} a_n \neq 0$, the 1^{st} criteria would fail from Theorem 9.14.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{n^2+5}$$

$$= \lim_{n\to\infty} \left(\frac{n^2}{n^2+5}\right)\left(\frac{\frac{1}{n^2}}{\frac{1}{n^2}}\right)$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{5}{n^2}}$$

$$= \frac{1}{1}$$

$$\lim_{n\to\infty} a_n = 1$$

Therefore, by the N-th Term Test, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^2+5}$ diverges.

$\boxed{9.5}$

Example! Consider $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

Earlier, we saw that this series was convergent. Can we approximate its sum?

Consider $S_{10} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10}$

$$S_{10} = \frac{1627}{2520}$$

$$S_{10} \approx 0.64563$$

We can say $S_{10} \approx S$, where $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

But, what is the error in our approximation?

$$S = S_N + R_N \quad \& \quad R_N = S - S_N$$

where R_N is the remainder

Theorem 9.15: Alternating Series Remainder

If a convergent alternating series satisfies the condition $a_{n+1} \leq a_n$, then the absolute value of the remainder R_N involved in approximating the sum S by S_N is less than (or equal to) the first neglected term. That is,

$$|S - S_N| = |R_N| \leq a_{N+1}$$

Proof: Let $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ be a convergent Alternating Series satisfying ① $\lim_{n \to \infty} a_n = 0$ and ② $a_{n+1} \leq a_n$.

$\boxed{9.5}$ Proof 9.15 cont'd

Consider $R_N = S - S_N$, $a_n \geq 0$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} a_n - \sum_{n=1}^{N} (-1)^{n+1} a_n$$

$$= (-1)^{N+2} a_{N+1} + (-1)^{N+3} a_{N+2} + (-1)^{N+4} a_{N+3} + (-1)^{N+5} a_{N+4} + \cdots$$

$$= (-1)^{N+2} \left[a_{N+1} + (-1)^1 a_{N+2} + (-1)^2 a_{N+3} + (-1)^3 a_{N+4} + \cdots \right]$$

$$|R_N| = \left| (-1)^{N+2} \cdot [a_{N+1} - a_{N+2} + a_{N+3} - a_{N+4} + \cdots] \right|$$

$$|R_N| = a_{N+1} - a_{N+2} + a_{N+3} - a_{N+4} + \cdots$$

$$= a_{N+1} - (a_{N+2} - a_{N+3}) - (a_{N+4} - a_{N+5}) + \cdots \leq a_{N+1}$$

So,

$$|R_N| \leq a_{N+1}.$$

Back to $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

with $S_{10} \approx S$,

and $S \approx 0.64563$

we'll have $R_{10} = \frac{1}{11} - \frac{1}{12} + \frac{1}{13} - \frac{1}{14} + \frac{1}{15} - \cdots$

so

$$R_{10} = \frac{1}{11} - \left(\frac{1}{12} - \frac{1}{13}\right) - \left(\frac{1}{14} - \frac{1}{15}\right) - \cdots \leq \frac{1}{11}$$

this means tha S_{10} is within $\frac{1}{11}$ of S.

or, $|S - S_{10}| \leq \frac{1}{11}$

$$-\frac{1}{11} \leq S - 0.64563 \leq \frac{1}{11}$$

$$0.64563 - \frac{1}{11} \leq S \leq \frac{1}{11} + 0.64563$$

$$\boxed{0.64563 \leq S \leq 0.73654}$$

$\boxed{9.5}$ (a) use Theorem 9.15 to determine the number of terms required to
#40 approximate the sum of the convergent series with an error of less than 0.001, & (b) use a graphing utility to approximate the sum of the series with an error less than 0.001.

consider $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} = \cos(1)$

$\cos(1) \approx 0.5403023059$

(a) The approximation "error" is the remainder. By Theorem 9.15 we have

$|R_N| \leq a_{N+1}$, with $a_n = \frac{1}{(2n)!}$

so $|R_N| \leq a_{N+1} = \frac{1}{[2(N+1)]!} < 0.001$

If we solve for N, we can find the number of terms we need to know S_N is within 0.001 of S.

Solve for N: $\frac{1}{(2N+2)!} < \frac{1}{1{,}000}$

$N=1,\quad \frac{1}{(2+2)!} = \frac{1}{4!} = \frac{1}{1\cdot2\cdot3\cdot4} = \frac{1}{24}$

$N=2,\quad \frac{1}{(4+2)!} = \frac{1}{6!} = \frac{1}{1\cdot2\cdot3\cdot4\cdot5\cdot6} = \frac{1}{720}$

$N=3,\quad \frac{1}{(6+2)!} = \frac{1}{8!} = \frac{1}{1\cdot2\cdot3\cdot4\cdot5\cdot7\cdot8} = \frac{1}{40{,}320}$

So, $\frac{1}{[2(N+1)]!} < 0.001$ when $N \geq 3$.

Since we start with $n=0$ in $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$, we'll
(a) use 4 terms when $N=3$ for S_3.

$\boxed{9.5}$ #40 cont'd

$$S_3 = \frac{(-1)^0}{[2(0)]!} + \frac{(-1)^1}{[2(1)]!} + \frac{(-1)^2}{[2(2)]!} + \frac{(-1)^3}{[2(3)]!}$$

$$S_3 = \frac{1}{0!} + \frac{(-1)}{2!} + \frac{1}{4!} + \frac{(-1)}{6!}$$

$$S_3 = 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720}$$

$$S_3 = \frac{389}{720}$$

(b) $S_3 = 0.5402\overline{7}$, or $S_3 \approx 0.5403$

Theorem 9.16 : Absolute Convergence

If the series $\sum |a_n|$ converges, then the series $\sum a_n$ converges.

Proof: If $\sum |a_n|$ converges, then $\sum 2 \cdot |a_n|$ converges.

For all n, we can see that

$$0 \le a_n + |a_n| \le 2|a_n|.$$

So, by Theorem 9.12, the Direct Comparison Test,

$\sum (a_n + |a_n|)$ converges because

$\sum 2|a_n|$ converges.

By writing $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} (a_n + |a_n|) - \sum\limits_{n=1}^{\infty} |a_n|$

We can see that the series $\sum\limits_{n=1}^{\infty} a_n$ converges,

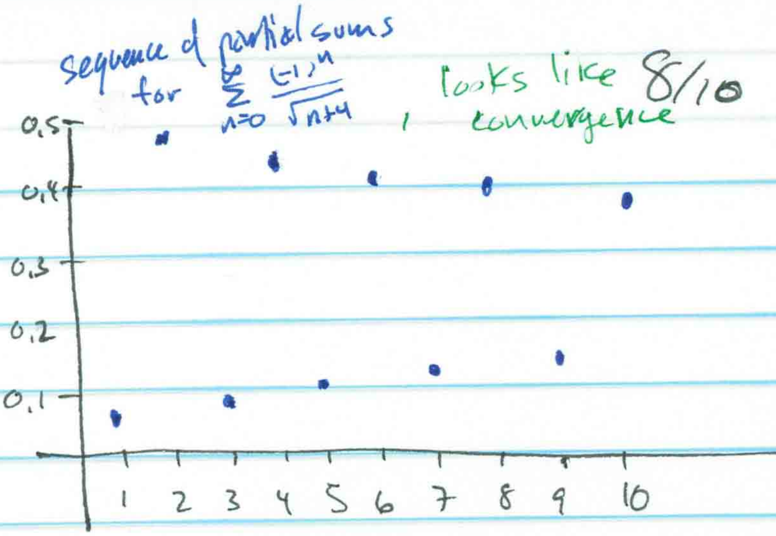
since both $\sum\limits_{n=1}^{\infty} (a_n + |a_n|)$ and $\sum\limits_{n=1}^{\infty} |a_n|$ are convergent series.

Definitions:

① $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges

② $\sum a_n$ is conditionally convergent if $\sum a_n$ converges but $\sum |a_n|$ diverges.

$\boxed{95}$



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#58 Consider $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+4}}$

① $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+4}}$

$= 0$ ✓

② Show $a_{n+1} \le a_n$

$$a_{n+1} = \frac{1}{\sqrt{(n+1)+4}} \quad , \quad a_n = \frac{1}{\sqrt{n+4}}$$

$$(n+1) \ge n$$
$$(n+1)+4 \ge n+4$$
$$\sqrt{(n+1)+4} \ge \sqrt{n+4}$$
$$\frac{1}{\sqrt{(n+1)+4}} \le \frac{1}{\sqrt{n+4}}$$

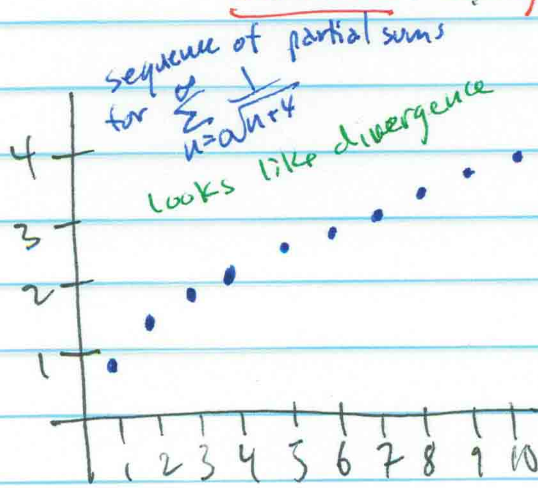
So, $a_{n+1} \le a_n$ ✓

By the Alternating Series Test, $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+4}}$ converges.

Does $\displaystyle\sum_{n=0}^{\infty} \left| \frac{(-1)^n}{\sqrt{n+4}} \right|$ converge? ← will answer the question of conditional convergence.

$$\sum_{n=0}^{\infty} \left| \frac{(-1)^n}{\sqrt{n+4}} \right| = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+4}}$$

TRY TO Show $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+4}}$ is divergent. use the Limit Comparison test with a p-series, $p = \frac{1}{2}$



$\boxed{9.5}$ ~~#58~~ cont'd $\frac{9}{10}$

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+4}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+3}} \quad \& \quad \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.$$

Divergent p-series
$p = 1/2$

$$\frac{1}{\sqrt{n+3}} \geq 0 \quad \& \quad 0 \leq \frac{1}{\sqrt{n}}$$

Evaluate: $\lim_{n\to\infty} \left(\frac{\frac{1}{\sqrt{n+3}}}{\frac{1}{\sqrt{n}}} \right) = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+3}}$

$$= \lim_{n\to\infty} \left(\frac{\sqrt{n}}{\sqrt{n+3}} \right) \left(\frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} \right)$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{3}{n}}}$$

$$= \frac{1}{\sqrt{1+0}}$$

$= 1$ ← Finite & positive!

By the Limit Comparison Test, both series diverge.

Since $\sum_{n=0}^{\infty} \left| \frac{(-1)^n}{\sqrt{n+4}} \right|$ diverges,
we can say that $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+4}}$ converges conditionally.

#50 consider $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{n}}$ ← "looks" like $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

convergent p-series

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① $\lim_{n\to\infty} a_n = \lim_{n\to} \frac{1}{n\sqrt{n}}$

$= 0$ ✓

9.5

$\frac{10}{10}$

#50 cont'd

(2) show $a_{n+1} \le a_n$

$$a_{n+1} = \frac{1}{(n+1)\sqrt{n+1}}, \quad a_n = \frac{1}{n\sqrt{n}}$$

$$n+1 \ge n$$

$$(n+1)\sqrt{n+1} \ge n\sqrt{n}$$

$$\frac{1}{(n+1)\sqrt{n+1}} \le \frac{1}{n\sqrt{n}}$$

So, $a_{n+1} \le a_n$ ✓

By the Alternating Series Test, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{n}}$ converges.

Does $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n\sqrt{n}}\right|$ converge? Yes!!

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

which is a convergent p-series with $p = 3/2$.

This means that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{n}}$ converges absolutely.