$\boxed{9.6}$ Ratio & Root Tests -

Theorem 9.17 - Ratio Test

Let $\sum a_n$ be a series with nonzero terms.

① $\sum a_n$ converges absolutely if $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$.

② $\sum a_n$ diverges if $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, or $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$

③ The Ratio Test is inconclusive if $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$.

#8 Consider $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cdot 4}{(2n)!}$

"looks" convergent
$(-1)^{n-1} \leftarrow$ Alternating
$\frac{4}{(2n)!} \to 0$ fast!

Let $a_n = \frac{4}{(2n)!}$ and $a_{n+1} = \frac{4}{[2(n+1)]!}$

Evaluate: $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\frac{\frac{4}{[2(n+1)]!}}{\frac{4}{(2n)!}}\right|$

$$= \lim_{n\to\infty} \left|\frac{4}{(2n+2)!} \cdot \frac{(2n)!}{4}\right|$$

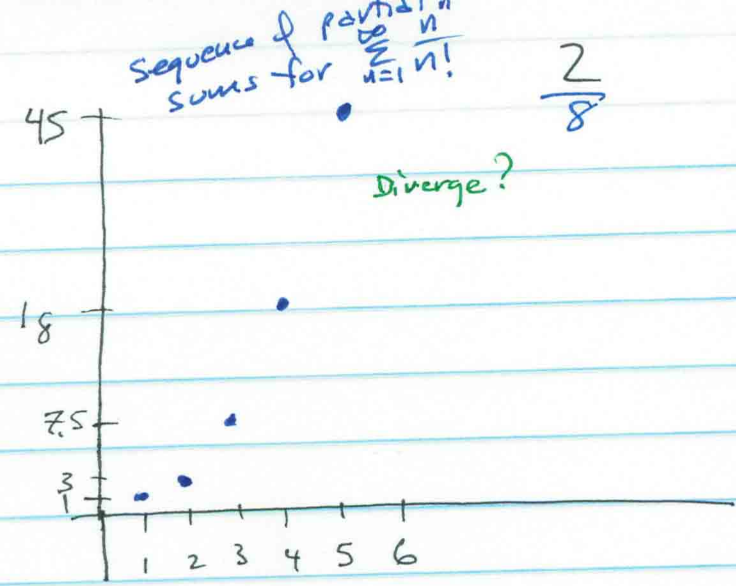
$$= \lim_{n\to\infty} \frac{(2n)!}{(2n+2)(2n+1)\cdot(2n)!}$$

$$= \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)}$$

$$= 0 \checkmark$$

Since $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$, the Ratio Test tells us that $\sum \frac{(-1)^{n-1}4}{(2n)!}$ converges absolutely.

$\boxed{9.6}$



#26 Consider $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Let $a_n = \frac{n^n}{n!}$ & $a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$

$a_n = \frac{n^n}{n!} > 0$

Evaluate: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$= \lim_{n\to\infty} \left| \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} \right|$$

$$= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}$$

$$= \lim_{n\to\infty} \frac{\cancel{(n+1)}\cdot(n+1)^n \cdot \cancel{n!}}{\cancel{(n+1)}\cdot \cancel{n!} \cdot n^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^n}{n^n}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$$

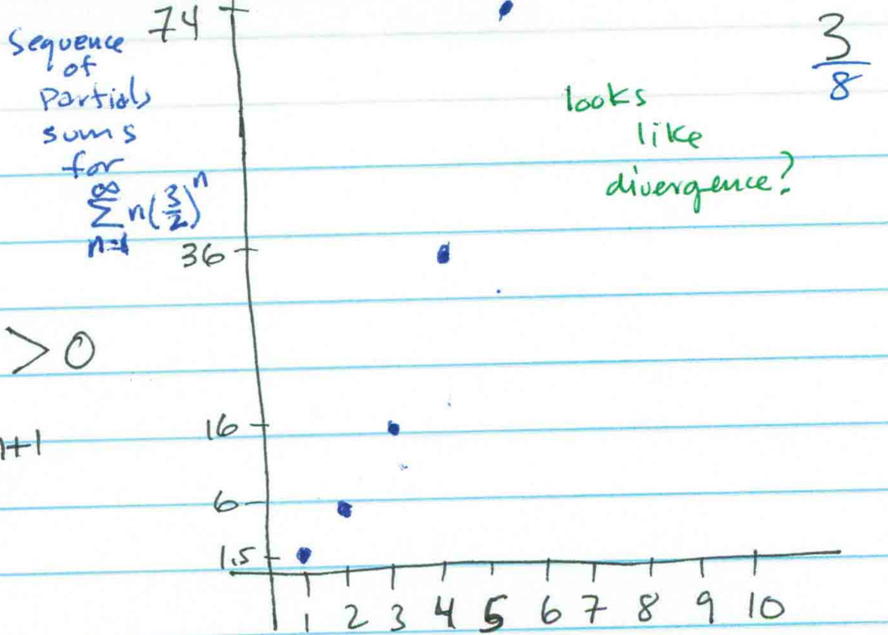
$= e$. Since $e > 1$, the Ratio Test, Theorem 9.17, says that $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges.

$(n+1)! = (n+1)(n)(n-1)(n-2)\cdots$
$= (n+1)\cdot n!$

$(n+1)^{n+1} = (n+1)^n \cdot (n+1)$

$\boxed{9.6}$

#16 Consider $\sum_{n=1}^{\infty} n\left(\frac{3}{2}\right)^n$



let $a_n = n\left(\frac{3}{2}\right)^n > 0$

&

$a_{n+1} = (n+1)\left(\frac{3}{2}\right)^{n+1}$

Evaluate: $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

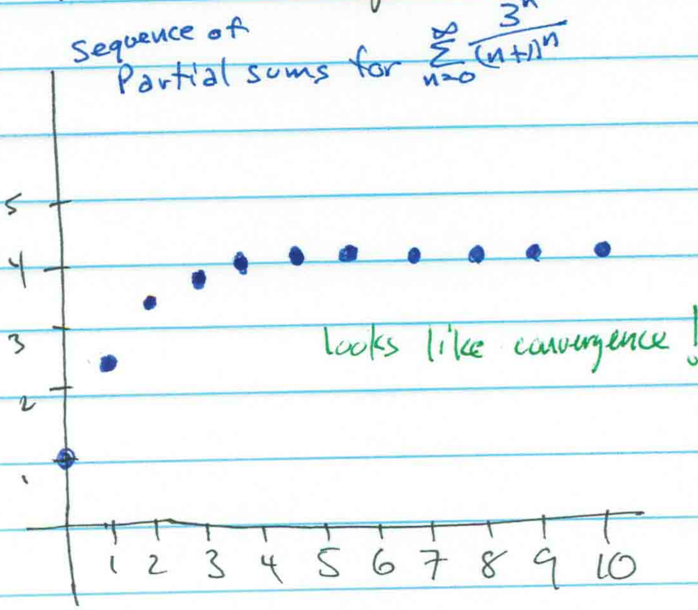
$$= \lim_{n\to\infty} \left| \frac{(n+1)\left(\frac{3}{2}\right)^{n+1}}{1} \cdot \frac{1}{n\cdot\left(\frac{3}{2}\right)^n} \right|$$

$$= \lim_{n\to\infty} \frac{(n+1)\cdot 3^{n+1}\cdot 2^n}{n\cdot 2^{n+1}\cdot 3^n}$$

$$= \lim_{n\to\infty} \frac{3\cdot(n+1)}{2\cdot n}$$

$= \frac{3}{2}$. Since $\frac{3}{2} > 1$, by Theorem 9.17, the Ratio Test, we can see that $\sum_{n=1}^{\infty} n\left(\frac{3}{2}\right)^n$ diverges.

#27 Consider $\sum_{n=0}^{\infty} \frac{3^n}{(n+1)^n}$



let $a_n = \frac{3^n}{(n+1)^n} > 0$

$a_{n+1} = \frac{3^{n+1}}{(n+2)^{n+1}}$

$\boxed{9.6}$

#27 cont'd ~~#27 cont'd~~ Evaluate: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}}{(n+2)^{n+1}}\cdot\frac{(n+1)^n}{3^n}\right|$

$= \lim_{n\to\infty}\frac{3\cdot(n+1)^n}{(n+2)\cdot(n+2)^n}$

How?

$= \lim_{n\to\infty}\left(\frac{3}{n+2}\right)\cdot\lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^n$ ☆☆☆

$= 0\cdot\left(\frac{1}{e}\right)$

$= 0$. Since $0<1$, by Theorem 9.17, the Ratio Test, we can see that $\sum_{n=0}^{\infty}\frac{3^n}{(n+1)^n}$ converges.

☆☆☆

$\lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^n = \frac{1}{e}$??

let $y = \lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^n$

$\ln(y) = \ln\left[\lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^n\right]$

$\ln(y) = \lim_{n\to\infty}\left(\ln\left[\left(\frac{n+1}{n+2}\right)^n\right]\right)$

$\ln(y) = \lim_{n\to\infty}\left[n\cdot\ln\left(\frac{n+1}{n+2}\right)\right]$

$\ln(y) = \lim_{n\to\infty}\left[\frac{\ln\left(\frac{n+1}{n+2}\right)}{\frac{1}{n}}\right]$

$\boxed{\begin{array}{l} \xrightarrow{\quad} \infty \quad n\to\infty \\ \ln\left(\frac{n+1}{n+2}\right)\approx\ln(1)\to \dfrac{0}{0} \\ \frac{1}{n}\to 0 \qquad \text{Indeterminate Form} \\ \text{use L'Hopital's Rule} \end{array}}$

$\ln(y) = \lim_{n\to\infty}\left[\frac{\ln(n+1)-\ln(n+2)}{\frac{1}{n}}\right]$

$\ln(y) = \lim_{n\to\infty}\frac{\frac{d}{dn}\left[\ln(n+1)-\ln(n+2)\right]}{\frac{d}{dn}\left(n^{-1}\right)}$

$\ln(y) = \lim_{n\to\infty}\left[\frac{\frac{1}{n+1}-\frac{1}{n+2}}{-1\cdot n^{-2}}\right]$

$\boxed{9.6}$ #27 cont'd

$$\ln(y) = \lim_{n\to\infty} \left[\frac{1}{n+1} - \frac{1}{n+2}\right]\cdot\left[\frac{n^2}{-1}\right]$$

gymnastics with limits!

$$\ln(y) = \lim_{n\to\infty} \left[\frac{(n+2)-(n+1)}{(n+1)(n+2)}\right]\left[\frac{n^2}{-1}\right]$$

$$\ln(y) = \lim_{n\to\infty} \left[\frac{1}{n^2+3n+2}\right]\left[\frac{-n^2}{1}\right]$$

$$\ln(y) = \lim_{n\to\infty} \frac{-n^2}{n^2+3n+2}$$

$$\ln(y) = -1 \longrightarrow e^{\ln(y)} = e^{-1}$$

$$\boxed{y = \frac{1}{e}}$$

zoinks!

Theorem 9.18: The Root Test

Let Σa_n be a series

① Σa_n converges absolutely if $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$

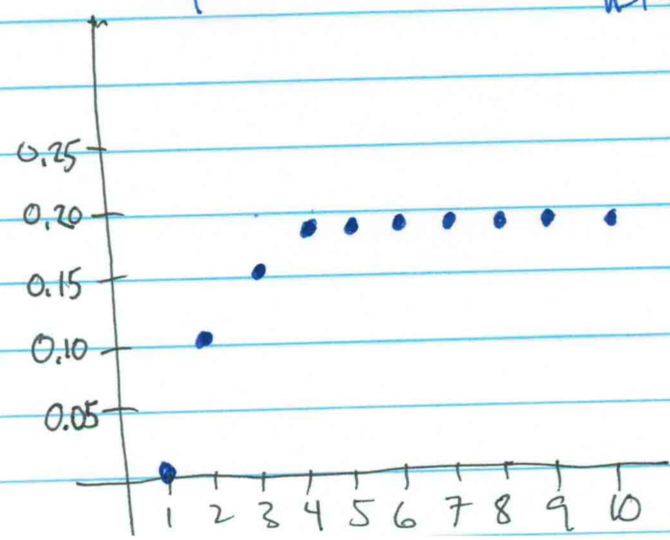
② Σa_n diverges if $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$

③ The Root Test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$.

#48 Consider $\sum_{n=1}^{\infty} \left[\frac{\ln(n)}{n}\right]^n$

Let $a_n = \left[\frac{\ln(n)}{n}\right]^n$

Sequence of partial sums for $\sum_{n=1}^{\infty} \left(\frac{\ln(n)}{n}\right.$



$\boxed{9.6}$ #48 cont'd

Evaluate: $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left[\frac{\ln(n)}{n}\right]^n\right|}$

$= \lim_{n\to\infty} \frac{\ln(n)}{n}$ → as $n\to\infty$ $\frac{\ln(n)}{n} \to \frac{\infty}{\infty}$

Indeterminate form!

Use L'Hopital's Rule!!

$= \lim_{n\to\infty} \frac{\frac{d}{dn}[\ln(n)]}{\frac{d}{dn}[n]}$

$= \lim_{n\to\infty} \frac{\frac{1}{n}}{1}$

$= \lim_{n\to\infty} \frac{1}{n}$

$= 0$. Since $0 < 1$, by Theorem 9.18, the Root Test, we can see that $\sum_{n=1}^{\infty} \left[\frac{\ln(n)}{n}\right]^n$ converges absolutely.

#42 Consider $\sum_{n=1}^{\infty} \left(\frac{-3n}{2n+1}\right)^{3n}$

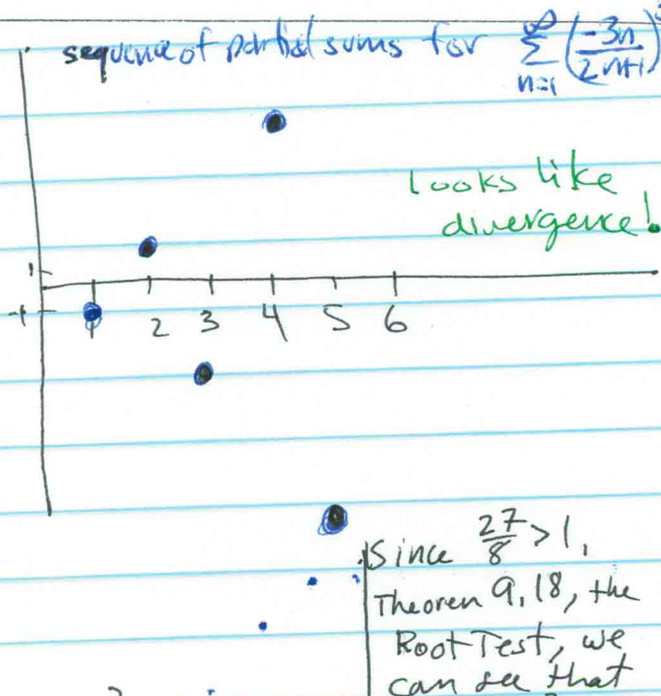
let $a_n = \left(\frac{-3n}{2n+1}\right)^{3n}$

Evaluate: $\lim_{n\to\infty} \sqrt[n]{|a_n|}$

$= \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{-3n}{2n+1}\right)^{3n}\right|}$

$= \lim_{n\to\infty} \left(\frac{3n}{2n+1}\right)^3$

$= \left(\lim_{n\to\infty} \frac{3n}{2n+1}\right)^3 = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$



Since $\frac{27}{8} > 1$, Theorem 9.18, the Root Test, we can see that $\sum_{n=1}^{\infty} \left(\frac{-3n}{2n+1}\right)^{3n}$ diverges!

$\frac{+}{8}$

9.6

Determine Convergence or divergence of each.
Choose the most efficient test.
(From Textbook page 643)
Summary on pg 644

Example 5

a. $\sum_{n=1}^{\infty} \frac{n+1}{3n+1}$

b. $\sum_{n=1}^{\infty} \left(\frac{\pi}{6}\right)^n$

c. $\sum_{n=1}^{\infty} n e^{-n^2}$

d. $\sum_{n=1}^{\infty} \frac{1}{3n+1}$

e. $\sum_{n=1}^{\infty} (-1)^n \left(\frac{3}{4n+1}\right)$

f. $\sum_{n=1}^{\infty} \frac{n!}{10^n}$

g. $\sum_{n=1}^{\infty} \left(\frac{n+1}{2n+1}\right)^n$

#81 Consider $\sum_{n=1}^{\infty} a_n$ where $a_1 = \frac{1}{3}$ and $a_{n+1} = \left(1+\frac{1}{n}\right)a_n$

Recursive definition

TRY The Ratio Test? $\longrightarrow$ * $\frac{a_{n+1}}{a_n} = \frac{\left(1+\frac{1}{n}\right)a_n}{a_n}$ *

we can see that $a_n > 0$ for all $n \geq 1$

Evaluate: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\left(1+\frac{1}{n}\right)\cdot \cancel{a_n}}{\cancel{a_n}}\right|$

$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)$

$= 1$ $\leftarrow$ The Ratio Test is Inconclusive!!

$\boxed{9.6}$ #81 cont'd

consider $a_{n+1} = (1+\frac{1}{n})a_n$

$a_{n+1} = a_n + \frac{1}{n}\cdot a_n$

this tells us that

$a_{n+1} \geq a_n$ for all $n \geq 1$

So, a_n is an increasing sequence.

Hence, $\lim_{n\to\infty} a_n \neq 0$

So, by the N-th Term Test, our Series $\sum_{n=1}^{\infty} a_n$ diverges.

Another way to see this: solve for a_n ??

$a_{n+1} = (1+\frac{1}{n})a_n$

$a_{n+1} = \left(\frac{n+1}{n}\right)a_n$

If $a_1 = \frac{1}{3}$, then $a_2 = \left(\frac{1+1}{1}\right)a_1$

$a_2 = 2a_1$

$a_3 = \left(\frac{2+1}{2}\right)a_2$

$a_3 = \frac{3}{2}a_2$

$a_4 = \frac{4}{3}a_3$

$a_5 = \frac{5}{4}a_4$, and so on....

$a_n = \left(\frac{n}{n-1}\right)\cdot a_{n-1}$

Consider the following product:

$a_2\cdot a_3\cdot a_4\cdot a_5 \cdots a_n = (2a_1)\left(\frac{3}{2}a_2\right)\left(\frac{4}{3}a_3\right)\left(\frac{5}{4}a_4\right)\cdots\left(\frac{n}{n-1}\right)a_{n-1}$

$a_2\cdot a_3\cdot a_4\cdot a_5 \cdots a_{n-1}\cdot a_n = a_1\cdot a_2\cdot a_3\cdot a_4 \cdot n\cdot a_{n-1}$

$a_n = a_1\cdot n$

$a_n = \frac{1}{3}\cdot n$ $\rightarrow$ $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{3}n$

Definitely Diverges 6!