$\boxed{9.7}$ Taylor Polynomials and Approximations

- Definition of n-th Taylor Polynomial AND n-th Maclaurin Polynomial

If f has n-derivatives at $(c, f(c))$, the the polynomial

$$P_n(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots$$

$$\cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n \leftarrow \text{called the n-th Taylor Polynomial of f at c.}$$

If $c=0$, then

$$P_n(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^n(0)}{n!}x^n$$

Is called the n-th Maclaurin polynomial of f.

Example: Find the 6^{th} degree Taylor Polynomial for $f(x) = \ln(1+x)$ centered at $c=0$

$f(c) = f(0) = \ln(1+0) = 0$

$f'(x) = \frac{1}{x+1} = (x+1)^{-1}$	$f'(0) = (0+1)^{-1} = 1$
$f''(x) = -1\cdot(x+1)^{-2}\cdot(1) = -(x+1)^{-2}$	$f''(0) = -(0+1)^{-2} = -1$
$f'''(x) = -2[-(x+1)^{-3}](1) = 2(x+1)^{-3}$	$f'''(0) = 2(0+1)^{-3} = 2$
$f^{iv}(x) = -3\cdot 2(x+1)^{-4}(1) = -6(x+1)^{-4}$	$f^{iv}(0) = -6(0+1)^{-4} = -6$
$f^{v}(x) = -4[-6(x+1)^{-5}] = 24(x+1)^{-5}$	$f^{v}(0) = 24(0+1)^{-5} = 24$
$f^{vi}(x) = -5\cdot 24(x+1)^{-6} = -120(x+1)^{-6}$	$f^{vi}(0) = -120(0+1)^{-6} = -120$

$$P_6(x) = f(0) + \frac{f'(0)}{1!}\cdot(x-0) + \frac{f''(0)}{2!}\cdot(x-0)^2 + \frac{f'''(0)}{3!}\cdot(x-0)^3 +$$

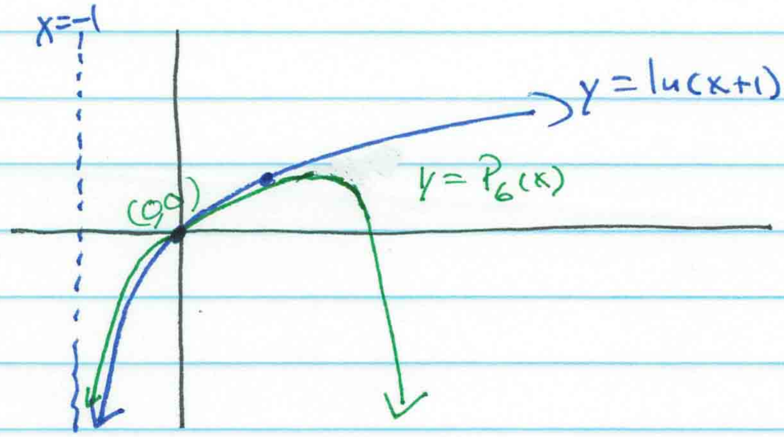
$$\frac{f^{iv}(0)}{4!}\cdot(x-0)^4 + \frac{f^{v}(0)}{5!}(x-0)^5 + \frac{f^{vi}(0)}{6!}(x-0)^6$$

9.7 cont'd

$$P_6(x) = 0 + \frac{(1)}{1!}x + \frac{(-1)}{2!}x^2 + \frac{(2)}{3!}x^3 + \frac{(-6)}{4!}x^4 + \frac{(24)}{5!}x^5 + \frac{(-120)}{6!}x^6$$

$$P_6(x) = x - \frac{1}{2}x^2 + \frac{2}{1\cdot2\cdot3}x^3 - \frac{6}{1\cdot2\cdot3\cdot4}x^4 + \frac{24}{1\cdot2\cdot3\cdot4\cdot5}x^5 - \frac{120}{1\cdot2\cdot3\cdot4\cdot5\cdot6}x^6$$

$$P_6(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6$$ ← This also a Maclaurin Polynomial.



Example: Find a 6^{th} degree Taylor Polynomial for $f(x) = \ln(x+1)$ centered at $x=1$.

$f(1) = \ln(1+1) = \ln 2$

$f'(x) = (x+1)^{-1}$	$f'(1) = (1+1)^{-1} = 2^{-1} = 1/2$
$f''(x) = -(x+1)^{-2}$	$f''(1) = -(1+1)^{-2} = -2^{-2} = -\frac{1}{4}$
$f'''(x) = 2(x+1)^{-3}$	$f'''(1) = 2(1+1)^{-3} = 2\cdot2^{-3} = 2/8 = \frac{1}{4}$
$f^{iv}(x) = -6(x+1)^{-4}$	$f^{iv}(1) = -6(1+1)^{-4} = -6\cdot2^{-4} = \frac{-6}{16} = -\frac{3}{8}$
$f^{v}(x) = 24(x+1)^{-5}$	$f^{v}(1) = 24(1+1)^{-5} = 24\cdot2^{-5} = \frac{24}{32} = \frac{3}{4}$
$f^{vi}(x) = -120(x+1)^{-6}$	$f^{vi}(1) = -120(1+1)^{-6} = -120\cdot2^{-6} = \frac{-120}{64} = \frac{-15}{8}$

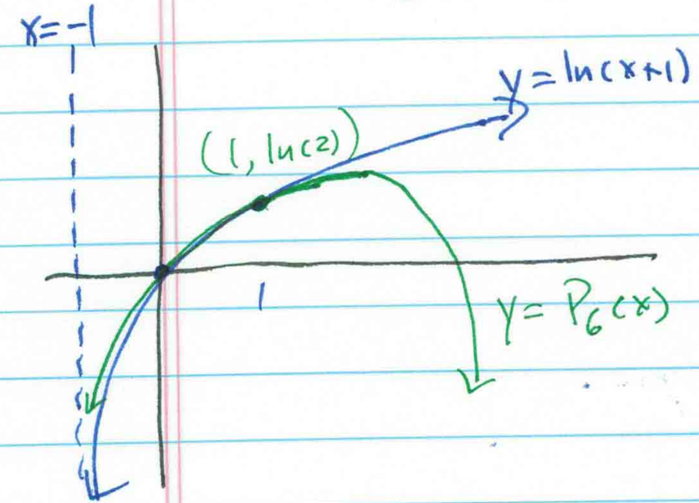
$$P_6(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{iv}(1)}{4!}(x-1)^4 + \frac{f^{v}(1)}{5!}(x-1)^5 + \frac{f^{vi}(1)}{6!}(x-1)^6$$

$\boxed{9.7}$

cont'd

$$P_6(x) = \ln 2 + \frac{\left(\frac{1}{2}\right)}{1!}(x-1) + \frac{\left(-\frac{1}{4}\right)}{2!}(x-1)^2 + \frac{\left(\frac{1}{4}\right)}{3!}(x-1)^3 + \frac{\left(-\frac{3}{8}\right)}{4!}(x-1)^4 + \frac{\left(\frac{3}{4}\right)}{5!}(x-1)^5 + \frac{\left(-\frac{15}{8}\right)}{6!}(x-1)^6$$

$$P_6(x) = \ln 2 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{24}(x-1)^3 - \frac{1}{64}(x-1)^4 + \frac{1}{160}(x-1)^5 - \frac{1}{384}(x-1)^6$$



$$\frac{-\frac{3}{8}}{4!} = \frac{-\cancel{3}}{8}\cdot\frac{1}{1\cdot 2\cdot \cancel{3}\cdot 4} = \frac{-1}{8\cdot 8} = \frac{-1}{64}$$

$$\frac{\frac{3}{4}}{5!} = \frac{\cancel{3}}{4}\cdot\frac{1}{2\cdot\cancel{3}\cdot 4\cdot 5} = \frac{1}{8\cdot 20} = \frac{1}{160}$$

$$\frac{-\frac{15}{8}}{6!} = \frac{-\cancel{15}}{8}\cdot\frac{1}{2\cdot\cancel{3}\cdot 4\cdot\cancel{5}\cdot 6} = \frac{-1}{16\cdot 24} = \frac{-1}{384}$$

#16 Find the Maclaurin Polynomial of degree 4 for $f(x) = e^{3x}$.

centered at $c=0$

$$f(0) = e^{3(0)} = e^{0} = 1$$

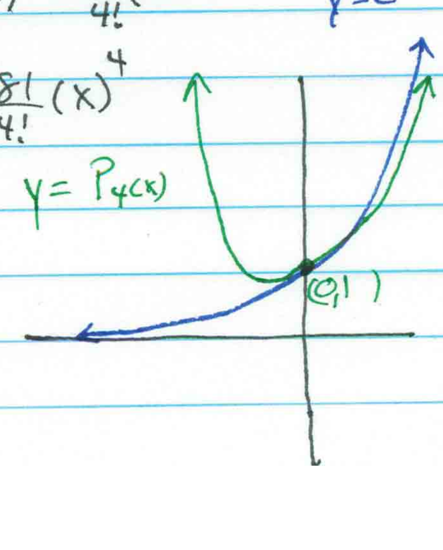
$$f'(x) = e^{3x}\cdot(3) = 3e^{3x}$$
$$f''(x) = 3e^{3x}(3) = 9e^{3x}$$
$$f'''(x) = 9e^{3x}(3) = 27e^{3x}$$
$$f^{iv}(x) = 27e^{3x}(3) = 81e^{3x}$$

$$f'(0) = 3\cdot e^{3(0)} = 3\cdot e^0 = 3$$
$$f''(0) = 9\cdot e^{3(0)} = 9\cdot e^0 = 9$$
$$f'''(0) = 27e^{3(0)} = 27\cdot e^0 = 27$$
$$f^{iv}(0) = 81\cdot e^{3x} = 81\cdot e^0 = 81$$

$$P_4(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{iv}(0)}{4!}(x-0)^4$$

$$P_4(x) = 1 + \frac{3}{1!}(x) + \frac{9}{2!}(x)^2 + \frac{27}{3!}(x)^3 + \frac{81}{4!}(x)^4$$

$$P_4(x) = 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4$$



$\boxed{9.7}$

Example: Use a 4^{th} degree Maclaurin polynomial to approximate $\ln(1.05)$.

If we let $f(x) = \ln(x+1)$, then we can use $P_4(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{IV}(0)}{4!}(x-0)^4$

From before, we know	
	$f(0) = \ln(0+1) = 0$
$f'(x) = \frac{1}{x+1} = (x+1)^{-1}$	$f'(0) = (0+1)^{-1} = 1$
$f''(x) = -1(x+1)^{-2}$	$f''(0) = -1(0+1)^{-2} = -1$
$f'''(x) = 2(x+1)^{-3}$	$f'''(0) = 2(0+1)^{-3} = 2$
$f^{IV}(x) = -6(x+1)^{-4}$	$f^{IV}(0) = -6(0+1)^{-4} = -6$

$$P_4(x) = 0 + \frac{(1)}{1!}(x) + \frac{(-1)}{2!}(x)^2 + \frac{(2)}{3!}(x)^3 + \frac{(-6)}{4!}(x)^4$$

$$P_4(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$$

$P_4(x) \approx \ln(x+1)$ centered at $c=0$.

$P_4(0.05) \approx \ln(0.05+1)$

$$P_4(0.05) = (0.05) - \frac{1}{2}(0.05)^2 + \frac{1}{3}(0.05)^3 - \frac{1}{4}(0.05)^4$$

$P_4(0.05) \approx 0.048790104167$

From a calculator, $\ln(1.05) \approx 0.048790164169$

So

$P_4(0.05)$ is a very close approximation of $\ln(1.05)$.

Is there a Theorem about this??

$\boxed{9.7}$

Theorem 9.19 - Taylor's Theorem

If a function f is differentiable through order $n+1$ in an interval I containing c, then, for each x in I, there exists z between x and c such that

$f(x) = P_n(x) + R_n(x)$

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x)$$

where $R_n(x) = \dfrac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$

Note:

$$|R_n(x)| \le \frac{|x-c|^{n+1}}{(n+1)!} \cdot \max|f^{(n+1)}(z)|$$

where $\max|f^{(n+1)}(z)|$ is the maximum value of $f^{(n+1)}(z)$ between x and c.

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pg 654 – "When applying Taylor's Theorem, you should not expect to be able to find the exact value of z. (If you could do this, approximation would not be necessary.) Rather, you try to find bounds for $f^{(n+1)}(z)$ from which you are able to tell how large the remainder $R_n(x)$ is."

Vocab: – $f(x) = P_n(x) + R_n(x)$

Exact value

Taylor Approximation Value

Remainder

– Error $= |R_n(x)| = |f(x) - P_n(x)|$

Error Associated with Approximation

$\boxed{9.7}$

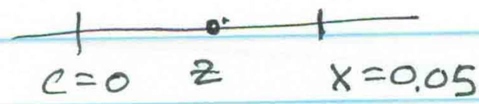
How good was our P_4 Maclaurin approximation of $\ln(1.05)$?

for $f(x) = \ln(x+1)$ we had

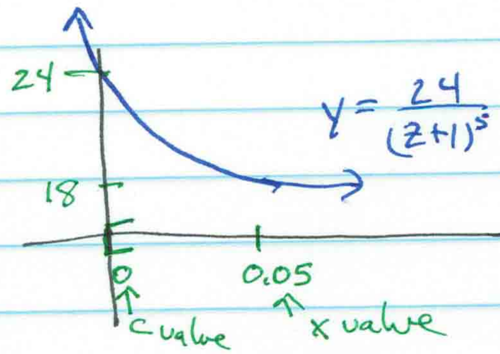
$$P_4(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$$

we know $f(x) = P_4(x) + R_4(x)$

$$R_4(x) = \frac{f^{(5)}(z)}{5!}(x-0)^5$$ with



$$f^{5}(x) = 24(x+1)^{-5}$$
$$f^{(5)}(z) = 24(z+1)^{-5}$$



on $[0, 0.05]$

$$\max|f^{(k+1)}(z)|$$
$$= \max|f^{(5)}(z)|$$
$$= \max\left|\frac{24}{(z+1)^5}\right| = \left|\frac{24}{(0+1)^5}\right|$$
$$\underline{= 24}$$

So, we have $|R_4(x)| \le \frac{|x-0|^5}{5!} \cdot \max|f^5(z)|$

$$|R_4(0.05)| \le \frac{|0.05-0|^5}{5!} \cdot 24$$

$$|R_4(0.05)| \le \frac{(0.05)^5 \cdot 24}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$$

$$|R_4(0.05)| \le \frac{(0.05)^5}{5} = \underline{6.25 \times 10^{-8}}$$

Bounds our error!

$\boxed{9.7}$ #52

Estimate $e^{0.3}$ with an error less than 0.001.

Determine the degree of this Maclaurin polynomial.

$(c=0)$ ← centered at zero.

Let $f(x) = e^x$

$$P_n(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \cdots + \frac{f^{n}(0)}{n!}(x-0)^n$$

$$f'(x) = e^x = f''(x) = f'''(x) = e^x \cdots = f(x)$$

$$f'(0) = e^0 = 1$$

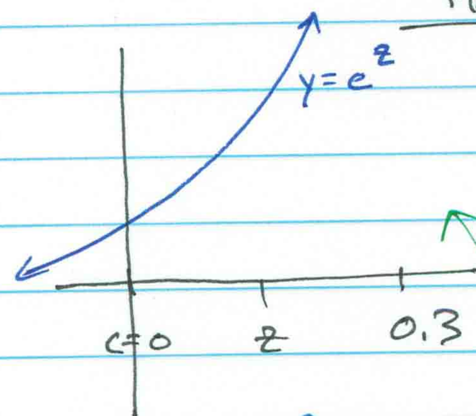
$$f''(0) = e^0 = 1, \text{ etc.} \ldots$$

$$\underline{f(0) = e^0 = 1}$$

$$P_n(x) = 1 + \frac{1}{1!}(x) + \frac{1}{2!}(x)^2 + \frac{1}{3!}(x)^3 + \cdots + \frac{1}{n!}(x)^n$$

← degree n Maclaurin Polynomial for $f(x) = e^x$

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-0)^{n+1}$$



$$R_n(x) = \frac{e^z}{(n+1)!}x^{n+1}$$

$$R_n(0.3) = \frac{e^z (0.3)^{n+1}}{(n+1)!}$$

All values are + so no absolute values

$$R_n(0.3) \le \frac{2(0.3)^{n+1}}{(n+1)!}$$

Since e^z increases we can estimate $e^{0.3} \le 2$

on $[0, 0.3]$

$\max |f^{(n+1)}(z)|$

$= \max |e^z|$

$= e^{0.3}$

≤ 2

Error ≤ 0.001

solve for n

$$\frac{2(0.3)^{n+1}}{(n+1)!} \le 0.001$$

$$\frac{2\left(\frac{3}{10}\right)^{n+1}}{(n+1)!} \le \frac{1}{1000}$$

$\boxed{9.7}$ #52 cont'd

continue to solve by using reciprocals

8/10

$$\frac{(n+1)!}{2}\left(\frac{10}{3}\right)^{n+1} \geq 1000$$

$$(n+1)!\left(\frac{10}{3}\right)^{n+1} \geq 2000$$

If $n=3$, $4!\left(\frac{10}{3}\right)^{4} \geq 2000$

$24 \cdot \left(\frac{10{,}000}{3}\right) \geq 2000$

$71{,}111.\bar{1} \geq 2000$, TRUE!

This means that a Maclaurin polynomial of degree 3 will approximate $e^{0.3}$ within 0.001 error.

$P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3$

$P_3(0.3) = 1 + (0.3) + \frac{1}{2}(0.3)^2 + \frac{1}{3!}(0.3)^3$

$P_3(0.3) = 1.3495$

$P_3(0.3) \approx e^{0.3}$

$1.3495 \approx e^{0.3}$

from a calculator

$e^{0.3} \approx 1.34985880758$

#50 Determine the degree of the Maclaurin polynomial required for the error in the approximation of $\cos(0.1)$ to be less than 0.001.
$(c=0)$ ← centered at zero for Maclaurin polynomials

Let $f(x) = \cos(x)$, $f(0) = \cos(0) = 1$

$P_n(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \cdots + \frac{f^{n}(0)}{n!}(x-0)^n$

$f'(x) = -\sin(x)$, $f''(x) = -\cos x$, $f'''(x) = \sin x$, $f^{IV}(x) = \cos x$

$f(0)=1$, $f'(0) = 0$, $f''(0) = -1$, $f'''(0) = 0$, $f^{IV}(0) = 1$

$\boxed{9.7}$ ~~#50 cont'd~~

$P_n(x) = 1 + \frac{(0)}{1}(x) + \frac{(-1)}{2!}(x)^2 + \frac{(0)}{3!}(x)^3 + \frac{(1)}{4!}(x)^4 + \dots$

$P_n(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \dots$

$P_n(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + \frac{(-1)^n}{(2n)!}x^{2n}$

$R_n(x) = \frac{f^{n+1}(z)}{(n+1)!}(x-0)^{n+1} = \frac{f^{n+1}(z)\, x^{n+1}}{(n+1)!}$

$f^{n+1}(z)$ is either $\sin(z)$ or $\cos(z)$, depending on n.
When considering $\max|f^{n+1}(z)|$ it does not matter
because $\sin(z) \le 1$ & $\cos(z) \le 1$.
So, we know $\underline{\max|f^{n+1}(z)| = 1}$

This means that $R_n(0.1) \le \frac{1 \cdot (0.1)^{n+1}}{(n+1)!}$

all values are + so no need for absolute values

So, Solve for n: $\frac{(0.1)^{n+1}}{(n+1)!} \le 0.001$

$$\frac{\left(\frac{1}{10}\right)^{n+1}}{(n+1)!} \le \frac{1}{1000}$$

$$(n+1)!\left(\frac{10}{1}\right)^{n+1} \ge 1000$$

If $\underline{n=2}$, $3!\ 10^3 \ge 1000$

$6 \cdot 1000 \ge 1000$

$6000 \ge 1000$ TRUE!

So, $P_2(x) = 1 - \frac{1}{2}x^2$ will approximate $\cos(0.1)$ within 0.001.

$\boxed{9.7}$ #50 cont'd

$P_2(0.1) = 1 - \frac{1}{2}(0.1)^2$

$P_2(0.1) = 0.995$

$P_2(0.1) \approx \cos(0.1)$

$0.995 \approx \cos(0.1)$

from a calculator...

$\cos(0.1) \approx 0.995004165278$