9.8 Power Series

In the next two sections we'll see that several important function can be represented **exactly** by an infinite series.

Example:

$$f(x) = e^x$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

For each real number x, it can be shown that the infinite series on the right converges to the number e^x.

- Definition: If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

is called a power series.

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n + \cdots$$

is called a power series centered at c, where c is a constant.

Example:

(a) $$\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1} = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots$$

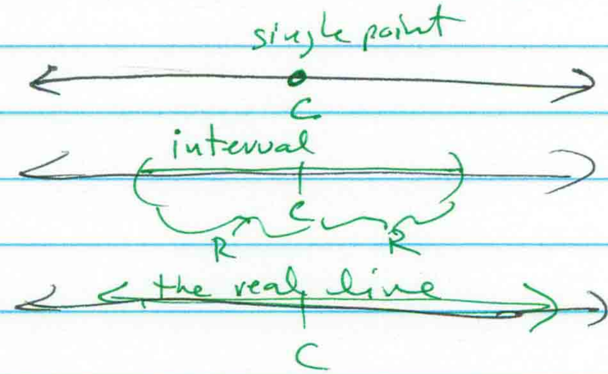
(b) $$\sum_{n=1}^{\infty} \frac{(2x)^n}{n^2} = 2x + \frac{4x^2}{4} + \frac{8x^3}{9} + \frac{16x^4}{16} + \frac{32x^5}{25} + \cdots$$

$$= 2x + x^2 + \frac{8x^3}{9} + x^4 + \frac{32}{25}x^5 + \cdots$$

9.8

We can view $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ as a function of x, where the <u>domain of f</u> is the set of all x for which the power series <u>converges</u>.
Therefore, we will need to know for which values of x the series converges.

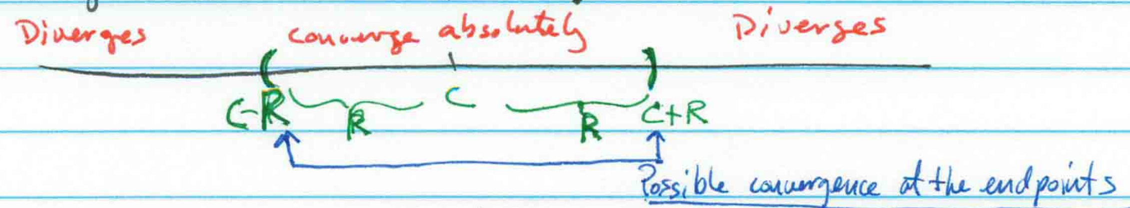
The domain of a power series has only three basic forms: a single point, an interval centered a c, or the entire real line.



<u>Theorem 9.20</u>: <u>Convergence of a Power Series</u>
For a power series centered at c, precisely one of the following is true.

1. The series converges only at c.

2. There exist a real number $R > 0$ such that the series converges absolutely for $|x-c| < R$, and diverges for $|x-c| > R$.



3. The series converges absolutely for all x.

$\boxed{9.8}$

The number R is the radius of convergence of the power series. If the series converges only at c, the radius of convergence is $R=0$, and if the series converges for all x, the radius of convergence is $R=\infty$. The set of all values of x for which the power series converges is the interval of convergence of the power series.

#16 Determine the interval of convergence of the series
$$\sum_{n=0}^{\infty} \frac{(3x)^n}{(2n)!}$$

To calculate R, we'll use the Ratio Test.

$a_n = \frac{(3x)^n}{(2n)!}$ is nonzero if $x \neq 0$.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(3x)^{n+1}}{[2(n+1)]!} \cdot \frac{(2n)!}{(3x)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(3x)\cdot(3x)^n \cdot (2n)!}{(2n+2)!\,(3x)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(2n)! \cdot 3x}{(2n+2)(2n+1)(2n)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{3x}{(2n+2)(2n+1)} \right|$$

$$= |3x| \cdot \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)}$$

$$= |3x| \cdot 0$$

$$= 0$$

Since $0 < 1$, by the Ratio Test, $\sum_{n=0}^{\infty} \frac{(3x)^n}{(2n)!}$ converges for all x. Therefore, the interval of convergence is $(-\infty, \infty)$.

#26 $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ $\qquad a_n = \dfrac{(-1)^n x^{2n+1}}{2n+1}$ is non-zero for $x \neq 0$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2(n+1)+1}}{2(n+1)+1}\cdot\frac{2n+1}{(-1)^n x^{2n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^{2n+3}}{2n+3}\cdot\frac{2n+1}{x^{2n+1}}\right|$$

$$= \lim_{n\to\infty} |x^2|\cdot\frac{2n+1}{2n+3}$$

$$= x^2\cdot\lim_{n\to\infty}\frac{2n+1}{2n+3}$$

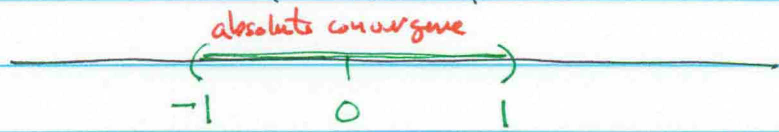
$$= x^2\cdot 1$$

$$= x^2$$

By the Ratio Test, we have convergence when $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, So, when $x^2<1$ we will have convergence.

Solve for x: $\quad x^2 < 1$

$$-1 < x < 1$$



what about the endpoints?

Let's plug-in $x=1$ and check!

If $x=1$, then $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n(1)^{2n+1}}{2n+1}$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$$

$\boxed{9.8}$

#26 cont'd

Use the Alternating Series Test to check the convergence of $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.

① $a_n = \frac{1}{2n+1}$, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2n+1}$

$= 0$ ✓

② check $0 < a_{n+1} \le a_n$

$$a_{n+1} = \frac{1}{2(n+1)+1} = \frac{1}{2n+3} \quad \Bigg| \quad a_n = \frac{1}{2n+1}$$

$$2n+3 \ge 2n+1$$

$$\frac{1}{2n+3} \le \frac{1}{2n+1} \quad \text{for all } n \ge 0.$$

By the A.S.T., $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ converges conditionally.

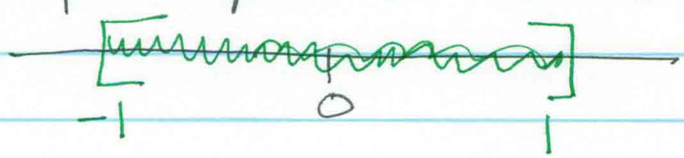
Plug-in $x = -1$: $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n (-1)^{2n+1}}{2n+1}$ ← always odd

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (-1)}{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$$

The A.S.T will tell us that $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$ will converge conditionally.

Since we have convergence at both $x=1$ & $x=-1$, the interval of convergence is $[-1, 1]$



u_n instead of a_n

$\boxed{9.B}$

#24 $\quad \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n\,2^n}$ $\quad$ let $u_n = \dfrac{(-1)^{n+1}(x-2)^n}{n\,2^n}$

$$u_{n+1} = \frac{(-1)^{(n+1)+1}(x-2)^{n+1}}{(n+1)\,2^{(n+1)}}$$

$$= \frac{(-1)^{n+2}(x-2)^{n+1}}{(n+1)\,2^{(n+1)}}$$

$$\lim_{n\to 1}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(x-2)^{n+1}}{(n+1)\,2^{(n+1)}}\cdot\frac{n\,2^n}{(-1)^{n+1}(x-2)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}\cdot n\,2^n}{(x-2)^n\cdot(n+1)2^{(n+1)}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-2)\cdot n}{(n+1)\cdot 2}\right|$$

$$= \frac{|x-2|}{2}\cdot\lim_{n\to\infty}\left(\frac{n}{n+1}\right)$$

$$= \frac{|x-2|}{2}\cdot 1$$

$$= \frac{|x-2|}{2}$$ $\leftarrow$ we have convergence when this limit is less than 1.

Solve for x: $\quad \dfrac{|x-2|}{2} < 1$

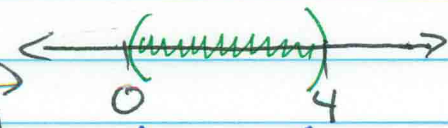
$$2\cdot\frac{|x-2|}{2} < 2\cdot 1$$

$$|x-2| < 2$$

$$-2 < x-2 < 2$$

$$-2+2 < 2+x-2 < 2+2$$

$$0 < x < 4$$



check endpoints!

9.8 #24 cont'd

check $x=4$: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}[(4)-2]^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}2^n}{n2^n}$

$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

We can use the A.S.T. to see that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges (conditionally).

check: $x=0$: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}[(0)-2]^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-2)^n}{n2^n}$

$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cdot(-1)^n 2^n}{n\cdot 2^n}$

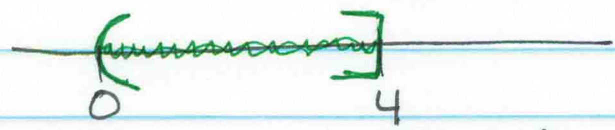
$= \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n}$ Always odd

$= \sum_{n=1}^{\infty} \frac{-1}{n}$

$= -\sum_{n=1}^{\infty} \frac{1}{n}$

This series diverges since it is a harmonic series.

Therefore, the interval of convergence is $(0,4]$.



Theorem 9.21: Properties of Functions Defined by Power Series

If the function $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ has a radius of convergence $R>0$, then, on the interval $(c-R, c+R)$, f is differentiable & continuous. The derivative and anti-derivative of f are as follows:

① $f'(x) = \sum_{n=1}^{\infty} n\cdot a_n (x-c)^{n-1}$

$= a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots$

② $\int f(x)dx = C + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1}$

$= C + a_0(x-c) + a_1\frac{(x-c)^2}{2} + a_2\frac{(x-c)^3}{3}$

9.8 **#45.** Find the interval of convergence for
(a) $f(x)$, (b) $f'(x)$, (c) $f''(x)$, and (d) $\int f(x)\,dx$.

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$$

$$f'(x) = \sum_{n=1}^{\infty} n \left(\frac{x}{2}\right)^{n-1} \cdot \left(\frac{1}{2}\right)$$ by chain Rule

$$\boxed{f'(x) = \sum_{n=1}^{\infty} \frac{n}{2} \left(\frac{x}{2}\right)^{n-1}}$$

by chain Rule

$$f''(x) = \sum_{n=2}^{\infty} \frac{n}{2} \cdot \left[(n-1) \cdot \left(\frac{x}{2}\right)^{n-2} \cdot \left(\frac{1}{2}\right)\right]$$

$$\boxed{f''(x) = \sum_{n=2}^{\infty} \left(\frac{n}{2}\right)\left(\frac{n-1}{2}\right)\left(\frac{x}{2}\right)^{n-2}}$$

let $z = \frac{x}{2}$

$\frac{dz}{dx} = \frac{1}{2}$

$\boxed{2\,dz = dx}$

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{n+1}}{n+1} \cdot (2)$$

$$\boxed{\int f(x)\,dx = C + \sum_{n=}^{\infty} \left(\frac{2}{n+1}\right)\left(\frac{x}{2}\right)^{n+1}}$$

The radius of convergence will be the same for all four functions. Only the intervals' endpoints may differ. __You__ supply the endpoint checks for each.

Find the radius of convergence: let $u_n = \left(\frac{x}{2}\right)^n$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\left(\frac{x}{2}\right)^{n+1} \cdot \left(\frac{2}{x}\right)^n\right|$$

$$= \lim_{n\to\infty} \left|\frac{x^{n+1} \cdot 2^n}{2^{n+1} \cdot x^n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{x}{2}\right|$$

$\boxed{9.8}$

#45 cont'd

We have convergence when

$$\lim_{n\to\infty} \left|\frac{x}{2}\right| < 1$$

$$\lim_{n\to\infty} \left|\frac{x}{2}\right| = \left|\frac{x}{2}\right| \cdot \lim_{n\to\infty} 1 = \left|\frac{x}{2}\right|$$

$$\left|\frac{x}{2}\right| < 1$$

So, solve for x!

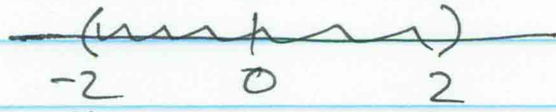
$$\left|\frac{x}{2}\right| < 1$$

$$2 \cdot \frac{|x|}{2} < 1 \cdot 2$$

$$|x| < 2$$

$$-2 < x < 2$$

Therefore, the radius of convergence is 2 units



check the endpoints!

#64 TRY This if you want: $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

(a) Find the interval of convergence.

(b) Show that $f'(x) = f(x)$

(c) Show that $f(0) = 1$

(d) Identify the function

(a) $$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{x^n \cdot x \cdot n!}{(n+1)(n!) \cdot x^n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{x}{n+1}\right|$$

$$= |x| \cdot \lim_{n\to\infty} \frac{1}{n+1}$$

$$= |x| \cdot 0 = 0$$

$R = \infty$

9.8 #64 cont'd

(b) $f'(x) = \sum_{n=1}^{\infty} \frac{n \cdot x^{n-1}}{n!}$

$= \sum_{n=1}^{\infty} \frac{n \cdot x^{n-1}}{n \cdot [(n-1)!]}$

$= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$ ← re-write for "n = 0" or use $K = n-1$

then if $n = 1$, we'll have $K = 1-1$ ← $K = 0$

$= \sum_{n=1}^{\infty} \frac{x^K}{K!}$

$= \sum_{K=0}^{\infty} \frac{x^K}{K!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$f'(x) = f(x)$ ✓

(c) $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$f(0) = 1 + \frac{(0)}{1!} + \frac{(0)^2}{2!} + \frac{(0)^3}{3!} + \cdots$

$f(0) = 1$ ✓

(d) since $f'(x) = f(x)$ we can say

$\frac{f'(x)}{f(x)} = 1$

$\frac{f'(x)}{f(x)} \cdot dx = 1 \cdot dx$

$\int \frac{f'(x)}{f(x)} dx = \int 1 dx$

$\int \frac{f'(x)}{u} \left(\frac{du}{f'(x)}\right) = x + C$

$\int \frac{1}{u} du = x + C$

$\ln|u| = x + C$

$\ln|f(x)| = x + C$

Let $u = f(x)$

$\frac{du}{dx} = f'(x)$

$du = f'(x) dx$

$\frac{du}{f'(x)} = dx$

$\boxed{9.8}$ #64 cont'd

$$e^{\ln|f(x)|} = e^{(x+c)}$$

$$|f(x)| = e^x \cdot e^c$$

$$|f(x)| = e^c \cdot e^x$$

$$f(x) = \underbrace{\pm e^c} \cdot e^x$$

let $D = \pm e^c$

So, $f(x) = De^x$

if $x = 0$, the $f(0) = 1$

and $De^{(0)} = D \cdot 1 = D$

$$f(0) = De^0$$

$$1 = D$$

Therefore, we have $f(x) = 1 \cdot e^x$

$$f(x) = e^x$$

AND $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ ✓ wild!!