MATH 155 - Chapter 8.5 - Partial Fractions
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Decomposition of $\dfrac{R(x)}{Q(x)}$ into Partial Fractions:

1. **Divide if Improper:** If $\dfrac{R(x)}{Q(x)}$ is an improper fraction (deg $(R) \geq$ deg (Q)), divide the denominator into the numerator by using long-division of polynomials to obtain
$$\frac{R(x)}{Q(x)} = f(x) + \frac{r(x)}{Q(x)}$$
where deg $r <$ deg Q.

2. **Factor Denominator:** Completely factor the denominator of $\dfrac{R(x)}{Q(x)}$ into irreducible factors.

Case 1. The denominator $Q(X)$ is a product of distinct linear factors:
Suppose $Q(x)$ factors into
$$Q(x) = (a_1x+b_1)(a_2x+b_2)(a_3x+b_3)\cdots\cdot(a_kx+b_k) \qquad \text{(No factor is repeated).}$$
Then
$$\frac{R(x)}{Q(x)} = \frac{A_1}{(a_1x+b_1)} + \frac{A_2}{(a_2x+b_2)} + \frac{A_3}{(a_3x+b_3)} + \cdots\cdot + \frac{A_k}{(a_kx+b_k)}.$$

Example: $\dfrac{1}{(2x+1)(3x-1)} = \dfrac{A}{2x+1} + \dfrac{B}{3x-1}$

Case 2. $Q(x)$ is a product of linear factors, some of which are repeated:
Suppose $Q(x)$ factors into
$$Q(x) = (ax+b)^k$$
Then
$$\frac{R(x)}{Q(x)} = \frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \cdots\cdot + \frac{A_k}{(ax+b)^k}.$$

Example: $\dfrac{1}{x^2(3x-1)^3} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{(3x-1)} + \dfrac{D}{(3x-1)^2} + \dfrac{E}{(3x-1)^3}$

Case 3. $Q(x)$ contains irreducible quadratic factors, none of which is repeated:
Suppose $Q(x)$ factors into

$$Q(x) = ax^2 + bx + c$$

Then

$$\frac{R(x)}{Q(x)} = \frac{Ax + B}{ax^2 + bx + C}$$

Example: $\dfrac{1}{(x^2+3)(x^3-1)} = \dfrac{Ax+B}{x^2+3} + \dfrac{Cx^2+Dx+E}{x^3-1}$

Case 4. $Q(x)$ contains a repeated irreducible quadratic factor:
Suppose $Q(x)$ factors into

$$Q(x) = (ax^2 + bx + c)^k$$

Then

$$\frac{R(x)}{Q(x)} = \frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \frac{A_3x + B_3}{(ax^2 + bx + c)^3} + \cdots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k}$$

Example: $\dfrac{1}{(x^2+x+1)(x^2+1)^3} = \dfrac{Ax+B}{x^2+x+1} + \dfrac{Cx+D}{x^2+1} + \dfrac{Ex+F}{(x^2+1)^2} + \dfrac{Gx+H}{(x^2+1)^3}$