MATH 155 - Chapter 9.10 - Taylor and Maclaurin series:
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1. **Theorem: Uniqueness Theorem**

   Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ represent a power series for all x in an open interval I containing a. Then

   $$a_n = \frac{f^{(n)}(a)}{n!}$$

   hence, $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$

2. **Definition: (Taylor Series and Maclaurin Series)** If a function f has derivatives of all orders at $x = c$, then the series

   $$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$
   $$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \cdots$$

   is called the **Taylor series for** $f(x)$ **at** a**.** Moreover, if $c = 0$, then the series is the **Maclaurin series for** f.

3. **Theorem: Convergence of Taylor Series**

   If $\lim_{n\to\infty} R_n = 0$ for all x in the interval I, then the Taylor series for f converges and equals $f(x)$,

   $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

4. **Theorem: Binomial Series**

For any real number k and for $|x| < 1$,

$$f(x) = (1+x)^k = 1 + C_1^k x + C_2^k x^2 + C_3^k x^3 + \cdots + C_k^k x^k + \cdots$$

where

$$C_n^k = \frac{k!}{n!(k-n)!} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

(we read C_n^k as "k choose n.")
Hence,

$$f(x) = (1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \frac{k(k-1)(k-2)(k-3)}{4!}x^4 + \cdots$$

The radius of convergence is $R = 1$, and hence, the interval of convergence is $(-1, 1)$.

5. **Guidelines for Finding a Taylor Series**

1. Differentiate $f(x)$ several times and evaluate each derivative a a. Try to recognize the pattern in these numbers.

2. Use the sequence developed in the first step to form the Taylor coefficients $a_n = \frac{f^{(n)}(a)}{n!}$ and determine the interval of convergence for the resulting power series.

3. Within this interval of convergence, determine whether or not the series converges to $f(x)$.