MATH 155 - Chapter 9.5 - Alternating Series; Absolute Convergence, and Conditional Convergence: (Can apply to positive and negative-term series)

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1. **Definition: (Alternating Series)** An **alternating series** is an infinite series of the form

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + a_5 - \cdots$$

or

$$\sum_{n=1}^{\infty}(-1)^{n}a_n = -a_1 + a_2 - a_3 + a_4 - a_5 + \cdots$$

where $a_n > 0$ for all n.

2. **Theorem: Alternating Series Test**

Given an alternating series of the form $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ or $\sum_{n=1}^{\infty}(-1)^{n}a_n$, let $a_n > 0$ for all n.

If

1. $a_n \geq a_{n+1} > 0$ for all n (ie. the sequence $\{a_n\}$ is decreasing for all n), and
2. $\lim_{n\to\infty} a_n = 0$.

Then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ or $\sum_{n=1}^{\infty}(-1)^{n}a_n$ converge.

Note: Suppose $\{a_n\}$ is decreasing for all n, but $\lim_{n\to\infty} a_n \neq 0$. That does NOT imply that $\sum_{n=1}^{\infty}(-1)^{n}a_n$ diverges!!

3. **Theorem: Alternating Series Remainder**

If a convergent alternating series satisfies the condition $a_n \geq a_{n+1}$ for all n, then the absolute value of the remainder R_N involved in approximating the sum S by S_N is less than or equal to the first neglected term. ie.

$$|R_N| = |S - S_N| \leq a_{N+1}$$

4. **Definition: (Absolute Convergence)** We say that the series $\sum_{n=1}^{\infty} a_n$ **converges absolutely** if $\sum_{n=1}^{\infty} |a_n|$ converges. (ie. the sum is a finite number.)

5. **Theorem: Absolute convergence Test**

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

6. **Definition: (Conditional Convergence)** A series that converges but does NOT absolutely converges is called **conditionally convergent.** ie. $\sum_{n=1}^{\infty} a_n$ converges conditionally, if $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ diverges.

NOTE: A series can either be

1. $\sum_{n=1}^{\infty} |a_n|$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges. (**Absolutely Convergent**)
2. $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges. (**Conditionally Convergent**)
3. $\sum_{n=1}^{\infty} a_n$ diverges. (**Divergent**)