MATH 155 - Chapter 9.7 - Taylor Polynomials and Approximations:

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1. **Definition: (Taylor Polynomial)** The n-th degree **Taylor Polynomial of the function f at the point $x = a$** is given by

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$
$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

2. **Definition: (Mclaurin Polynomial)** The n-th degree **Mclaurin Polynomial of the function f** is a Taylor Polynomial when $a = 0$, which is given by

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k$$
$$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x4 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

3. **Theorem: Taylor's Theorem (Taylor's Formula with Remainder):**

Let f be a function whose $(n+1)$-st derivative $f^{(n+1)}(x)$ exists for each x in an open interval I containing a. Then, for each x in I,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n(x)$$

ie.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

where the remainder (or error) $R_n(x)$ is given by the formula

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}, \text{ and } c \text{ is some point between } x \text{ and } a.$$

Furthermore,

$$|R_n(x)| = \left|\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}\right| \le \frac{|x-a|^{n+1}}{(n+1)!} \max |f^{(n+1)}(c)|$$

where c is some point between a and x.