MLE of Parameters in the Drifted Brownian Motion and Its Error

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Abstract

We consider the maximum likelihood estimator of the unknown parameter in a class of multi-dimensional diffusion processes. We obtain a precised bound for the error of this estimator, which tends to 0 exponentially fast.

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1 Introduction

Diffusion processes play important roles in the study of mathematical finance and other applications. In those applications we need to identify the models with unknown parameters (see P.Rao [5]). In this paper, we consider the maximum likelihood estimator of the unknown in the following d-dimensional diffusion process

$$dX_{i,t} = dW_{i,t} + \mu_i(X_t, t, \theta)dt, \qquad (i = 1, ..., d) \tag{1.1}$$

where θ is the an unknown parameter and W_t is a d-dimensional standard Wiener processes. Without losing generality we assume that $\theta \in \Theta$ where Θ is a hypercube in $R^{\tilde{d}}$ with radius less than K_0.

We assume that the realized sample x_t of $(X_t,\ 0 \le t \le T)$ can be observed continuously in this paper for simplicity.

Our result is more than a multi-dimensional extension of the previous results. In one dimensional case, we extend J.P.N. Bishwal's well-known result [1] by lifting his monotonity condition imposed on $\frac{\partial}{\partial\theta}\mu$ (see the Remark in [6]), and we extend [6] by lifting the boundedness assumption imposed on μ. Our conditions are

(i) $|\mu(x,t,\theta_1) - \mu(x,t,\theta_2)| \le K_1|\theta_1 - \theta_2|$, $|\mu(x,t_1,\theta) - \mu(x,t_2,\theta)| \le K_2|t_1 - t_2|$ and $|\mu(x_1,t,\theta) - \mu(x_2,t,\theta)| \le K_2|x_1 - x_2|$, where K_1 and K_2 are constants;

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(ii) Denote by $B(x,\epsilon)$ the ball centered at x with radius ϵ. There are positive constants $K_3 \leq 2^{-\frac{1}{2}}$ and $K_4 \leq 1$ such that $\forall \epsilon \leq K_3$ and $\forall \theta_1, \theta_2 \in \Theta$,

$$\int \chi_{\{z \in B(x,\epsilon);\ \frac{|\mu(z,t,\theta_1)-\mu(z,t,\theta_2)|}{|\theta_1-\theta_2|} \geq \epsilon\}} dz \geq K_4 \epsilon^d \qquad \forall x \tag{1.2}$$

2 Main Results

Denote by P_W the Wiener measure on $C[0,T]$ and by $P_\theta^T = P_{X_t|\theta}$ the probability measure induced by the coordinate process $\{X_t^\theta\}_{t \leq T}$ for the given θ. Then by Girsanov's theorem

$$\frac{dP_{X_t|\theta}}{dP_W} = \exp\{\sum_i \int_0^T \mu_i(X_t,t,\theta)dX_{i,t} - \frac{1}{2}\sum_i \int_0^T \mu_i^2(X_t,t,\theta)dt\}$$

We take the continuous version of the above random field in (θ, T).

The maximum likelihood estimator (MLE) of θ is denoted by $\hat{\theta}_T$ which is defined as in [5]

$$\hat{\theta}_T = arg \max_{\theta \in \Theta} \frac{dP_{X_t|\theta}}{dP_W}$$

From the discussion in the last section, it follows easily the existence of MLE when T is not too small. Although we do not have the uniqueness, the result of this paper shows also that the possible different MLEs are close together when T is not too small.

Theorem 2.1 *When $T > 1$, we have for all $0 < r < K_0$*

$$P_\theta^T\{|\hat{\theta}_T - \theta| > r\} \quad \leq \quad (2K_0^{\tilde{d}}T^{\frac{\tilde{d}}{2}} + K_8 T^{(\tilde{d}+\frac{3}{2})})\left\{1 - K_9 r^2\right\}^{\frac{1}{4\tilde{d}}[\frac{T}{K_3^2}]}$$

which tends to 0 ($T \to \infty$), where

$$K_6 = \{\frac{1}{e^{2\sqrt{d}K_2}\sqrt{2\pi}}\exp(-\frac{1}{2})\exp[-2\sqrt{d}K_2(\sqrt{\frac{2}{\pi}}\sqrt{\frac{e^{2\sqrt{d}K_2}-1}{2\sqrt{d}K_2}}+1) - \frac{(\sqrt{d}K_2)^2}{2}]\}^d,$$

$$K_8 = 2^{\tilde{d}+2}K_5[2^{\tilde{d}+1}(\frac{\tilde{d}+1}{2(\tilde{d}+2)^2})^{\tilde{d}+2}K_0^{\tilde{d}+2}K_1^{2(\tilde{d}+2)} + 2^{\tilde{d}+1}\frac{1}{(\tilde{d}+2)^{\tilde{d}+2}}a_{\tilde{d}+2}K_1^{\tilde{d}+2}]\tilde{d}^{\frac{1}{2}}$$

and

$$K_9 = (1 \wedge (K_4K_6))(\frac{1}{2})^{\frac{2+d}{2}}\frac{K_3^4}{2}\frac{\tilde{d}}{(\tilde{d}+2)^2}\exp\{-\frac{K_0^2K_3^4}{2}\frac{\tilde{d}}{(\tilde{d}+2)^2}\} \leq \frac{1}{2^{\frac{2+d}{2}}eK_0^2},$$

$a_{\tilde{d}+2}$ is the constant of the martingale moment inequality (3.7), K_5 is the Kolmogorov's constant in Lemma 3.2, which are all precisely computable and given in the literatures,

The statement of the above theorem is a little complicated because we want to show the precise function relation, which is necessary in its statistical application.

We get immediately from the above theorem a simpler version

Corollary 2.2 *For all $0 < r < K_0$,*

$$\lim_{T\to\infty}\frac{1}{T}\log\left(P_\theta^T\{|\hat{\theta}_T - \theta| > r\}\right) \leq -\frac{1}{4\tilde{d}K_3^2}|\log(1 - K_9 r^2)|.$$

3 Some preliminary results

Denote

$$M_T(\theta) = \frac{dP^T_{\theta+u}}{dP^T_\theta}. \tag{3.3}$$

then we have (see [5], p.46)

$$M_T(u) = \exp\{\sum_i \int_0^T g_{i,t}(u)dW_{i,t} - \frac{1}{2}\sum_i \int_0^T g^2_{i,t}(u)dt\} \tag{3.4}$$

where $g_{i,t}(u) = \mu_i(X_t, t, \theta+u) - \mu_i(X_t, t, \theta)$.

According to ([2], p.165), when $m > 1$,

$$E[|M_T|^{2m}] \leq \epsilon^m (\frac{(1+\epsilon)2^{1-m}}{2m-1} - 1)^{-1} E[< M >^m_T]$$

Take $\epsilon = (2m-1)2^{m-1}$, we get

$$E[|M_T|^{2m}] \leq (2m-1)^m 2^{m(m-1)} (\frac{(1+(2m-1)2^{m-1})2^{1-m}}{2m-1} - 1)^{-1} E[< M >^m_T] \tag{3.5}$$

Lemma 3.1 *Let $k \geq 2$ be an integer. Under the assumption of (i)-(ii), we have $E_\theta[M_T] = 1$ and*

$$E_\theta[M_T^{\frac{1}{k}}(u_1) - M_T^{\frac{1}{k}}(u_2)]^k \leq [2^{k-1}(\frac{k-1}{2k^2})^k K_1^{2k} K_0^k T^k + 2^{k-1}\frac{1}{k^k} a_k K_1^k T^{\frac{k}{2}}]|u_1 - u_2|^k \tag{3.6}$$

where $a_k = (k-1)^{\frac{k}{2}} 2^{\frac{k}{2}(\frac{k}{2}-1)} (\frac{(1+(k-1)2^{\frac{k}{2}-1})2^{1-\frac{k}{2}}}{k-1} - 1)^{-1}$.

Proof The first statement is from Girsanov's theorem as $g_{i,t}(u)$ is bounded for each fixed u.. Let us prove the inequality. Let $\theta_1 = \theta + u_1, \theta_2 = \theta + u_2$, and $\delta_{i,t} = \mu_i(X_t, t, \theta_2) - \mu_i(X_t, t, \theta_1)$. Denote $V_t = [\frac{dP^t_{\theta_2}}{dP^t_{\theta_1}}]^{\frac{1}{k}}$. Then

$$\begin{aligned} 1 - V_T &= 1 - \exp\{\frac{1}{k}\int_0^T \sum_i \delta_{i,t} dW^{\theta_1}_{i,t} - \frac{1}{2k}\int_0^T \sum_i \delta^2_{i,t} dt\} \\ &= \frac{k-1}{2k^2}\int_0^T \sum_i V_t \delta^2_{i,t} dt - \frac{1}{k}\int_0^T \sum_i V_t \delta_{i,t} dW^{\theta_1}_{i,t} \end{aligned}$$

where $W^{\theta_1}_t$ is a Brownian motion under $P^T_{\theta_1}$. By (3.5)

$$\begin{aligned} E^T_{\theta_1}[|\int_0^T \sum_i V_t \delta_{i,t} dW^{\theta_1}_{i,t}|^k] &\leq a_k E^T_{\theta_1}[|\int_0^T \sum_i V_t^2 \delta^2_{i,t} dt|^{\frac{k}{2}}] \\ &\leq a_k K_1^k |u_1 - u_2|^k T^{\frac{k}{2}} E^T_{\theta_1}[|\frac{1}{T}\int_0^T V_t^2 dt|^{\frac{k}{2}}] \\ &\leq a_k K_1^k |u_1 - u_2|^k T^{\frac{k}{2}} \frac{1}{T}\int_0^T E^T_{\theta_1}[|V_t^k|] dt \\ &= a_k K_1^k |u_1 - u_2|^k T^{\frac{k}{2}} \end{aligned} \tag{3.7}$$

Thus

$$
\begin{aligned}
& E_\theta^T[|M_T^{\frac{1}{k}}(u_1) - M_T^{\frac{1}{k}}(u_2)|^k] \\
= \ & E_\theta^T\Big\{|\Big[\frac{dP_{\theta_1}^T}{dP_\theta^T}\Big]^{\frac{1}{k}} - \Big[\frac{dP_{\theta_2}^T}{dP_\theta^T}\Big]^{\frac{1}{k}}|^k\Big\} \\
= \ & \int |\Big[\frac{dP_{\theta_1}^T}{dP_\theta^T}\Big]^{\frac{1}{k}} - \Big[\frac{dP_{\theta_2}^T}{dP_\theta^T}\Big]^{\frac{1}{k}}|^k dP_\theta^T \\
= \ & \int |\Big[\frac{dP_{\theta_1}^T}{dP_\theta^T}\Big]^{\frac{1}{k}} - \Big[\frac{dP_{\theta_2}^T}{dP_\theta^T}\Big]^{\frac{1}{k}}|^k \frac{dP_\theta^T}{dP_{\theta_1}^T} dP_{\theta_1}^T \\
= \ & \int |1 - \Big[\frac{dP_{\theta_2}^T}{dP_{\theta_1}^T}\Big]^{\frac{1}{k}}|^k dP_{\theta_1}^T \\
= \ & E_{\theta_1}^T[|1 - V_T|^k] \\
= \ & E_{\theta_1}^T[|\frac{k-1}{2k^2}\int_0^T \sum_i V_t\delta_{i,t}^2 dt - \frac{1}{k}\int_0^T \sum_i V_t\delta_{i,t} dW_{i,t}^{\theta_1}|^k] \\
\leq \ & 2^{k-1}E_{\theta_1}^T[|\frac{k-1}{2k^2}\int_0^T \sum_i V_t\delta_{i,t}^2 dt|^k] + 2^{k-1}E_{\theta_1}^T[|\frac{1}{k}\int_0^T \sum_i V_t\delta_{i,t} dW_{i,t}^{\theta_1}|^k] \\
\leq \ & 2^{k-1}(\frac{k-1}{2k^2})^k K_1^{2k}|u_1-u_2|^{2k}T^k E_{\theta_1}^T[|\frac{1}{T}\int_0^T V_t dt|^k] + 2^{k-1}\frac{T^{\frac{k}{2}}}{k^k}a_k K_1^k |u_1-u_2|^k \\
\leq \ & 2^{k-1}(\frac{k-1}{2k^2})^k K_1^{2k}|u_1-u_2|^{2k}T^k + 2^{k-1}\frac{1}{k^k}a_k K_1^k|u_1-u_2|^k T^{\frac{k}{2}} \qquad (3.8) \\
\leq \ & 2^{k-1}(\frac{k-1}{2k^2})^k K_1^{2k}K_0^k|u_1-u_2|^k T^k + 2^{k-1}\frac{1}{k^k}a_k K_1^k|u_1-u_2|^k T^{\frac{k}{2}}.
\end{aligned}
$$

Lemma 3.2 *(Kolmogorov's Criterion) Suppose that we are given positive constants* $\tilde{d},\ \gamma,\ \alpha,\ \beta$ *and a domain* $\Theta \subset R^{\tilde{d}}$. *Suppose* $\alpha > \tilde{d}$ *and* $0 < \beta < \frac{\alpha - \tilde{d}}{\gamma}$. *Then there is a constant* K_5 *such that for any random field* $\xi(\theta)$ $(\theta \in \Theta)$

$$E[\sup_{|\theta_1-\theta_2|\leq h}|\xi(\theta_1)-\xi(\theta_2)|^\gamma] \leq K_5 h^{\beta\gamma}\left[\sup_{|\theta_1-\theta_2|\leq h}\frac{E|\xi(\theta_1)-\xi(\theta_2)|^\gamma}{|\theta_1-\theta_2|^\alpha}\right]$$

where K_5 *is a precisely computable constant depending on* $k,\ K_1,\ a_k$ *and* $\tilde{d}$.

Proof. See [3], pp.31–35. Actually, if

$$E|\xi(\theta_1)-\xi(\theta_2)|^\gamma \leq C|\theta_1-\theta_2|^\alpha \qquad \forall \theta_1,\ \theta_2$$

then we get from [3] p.32:

$$\sup_{|\theta_1-\theta_2|\leq h}|\xi(\theta_1)-\xi(\theta_2)|^\gamma \leq \tilde{K}^\gamma h^{\beta\gamma}$$

where $\tilde{K}$ is a random variable, which is denoted as $4|q|(\sum_{N1=1}^\infty \Delta_N^\beta(X(\omega))$ on p.34 of [3], such that $E[\tilde{K}^\gamma]$ is bounded by a function of $\alpha,\ \beta$ and γ. Thus we get the Lemma by taking the expectation on both sides of the above inequality.

Lemma 3.3 *For $k \geq \tilde{d}+1$ and $\beta < \frac{k-\tilde{d}}{k}$,*

$$E_\theta^T\{\sup_{|u_1-u_2|\leq h}|M_T^{\frac{1}{k}}(u_1) - M_T^{\frac{1}{k}}(u_2)|^k\} \leq K_5[2^{k-1}(\frac{k-1}{2k^2})^k K_0^k K_1^{2k}T^k + 2^{k-1}\frac{1}{k^k}a_k K_1^k T^{\frac{k}{2}}]h^{\beta k}$$

Proof From Lemma 3.1,

$$E_\theta[M_T^{\frac{1}{k}}(u_1) - M_T^{\frac{1}{k}}(u_2)]^k \quad \leq \quad [2^{k-1}(\frac{k-1}{2k^2})^k K_0^k K_1^{2k}T^k + 2^{k-1}\frac{1}{k^k}a_k K_1^k T^{\frac{k}{2}}]|u_1-u_2|^k$$

Thus, we get the conclusion by Kolmogorov's criterion for $\gamma = \alpha = k$.

Lemma 3.4 *Under the conditions (i)–(ii),*

$$E_\theta^T[M_T^{\frac{1}{k}}(u)] \quad \leq \quad \{E_\theta^T[\exp\{-\frac{k-2}{k^2}\int_0^T|\mu(X_t,t,\theta+u)-\mu(X_t,t,\theta)|^2dt\}]\}^{\frac{1}{2}} \tag{3.9}$$

Proof By Hölder inequality,

$$\begin{aligned}
&E_\theta^T[M_T^{\frac{1}{k}}(u)] \\
&= E_\theta^T[\exp\{\frac{1}{k}\int_0^T\sum_i g_{i,t}(u)dW_{i,t} - \frac{1}{2k}\int_0^T|g_t(u)|^2dt\}] \\
&= E_\theta^T[\exp\{\frac{1}{k}\int_0^T\sum_i g_{i,t}(u)dW_{i,t} - \frac{1}{k^2}\int_0^T|g_t(u)|^2dt\}\exp\{-\frac{k-2}{2k^2}\int_0^T|g_t(u)|^2dt\}] \\
&\leq \{E_\theta^T[(\exp\{\frac{2}{k}\int_0^T\sum_i g_{i,t}(u)dW_{i,t} - \frac{2}{k^2}\int_0^T|g_t(u)|^2dt\})\}^{\frac{1}{2}}\{E_\theta^T[(\exp\{-\frac{k-2}{k^2}\int_0^T|g_t(u)|^2dt\})]\}^{\frac{1}{2}}
\end{aligned}$$

The first factor on the last line is less or equal to 1. So we get the Lemma.

Lemma 3.5 *Suppose that $b=(b_i)_i$ is a vector field such that*

$$|b^i(x,t)| \leq \varepsilon_i + \delta_i|x-y|, \qquad \forall x$$

for given y. Define

$$c_1 = \sqrt{\sum_{i=1}^d \varepsilon_i^2} \text{ and } c_2 = \sqrt{\sum_{i=1}^d \delta_i^2}$$

and define for every $q>0,\ t\geq 0$,

$$a_{1,q} = \frac{1}{\sqrt{2\pi}}\int_R |x|^q e^{-\frac{x^2}{2}}dx.$$

Let

$$J_1 = \left\{ \frac{1}{e^{2c_2}\sqrt{2\pi}} \exp\left(-\frac{1}{2}e^{2c_2}\right) \exp\left[-2c_1\left(a_{1,1}\sqrt{\frac{e^{2c_2}-1}{2c_2}}+1\right)-\frac{c_1^2}{2}\right]\right\}^d$$

and

$$J_2 = \left\{ \frac{e^{2c_2}}{\sqrt{2\pi}}[1 + c_1\left(\zeta_{\frac{3}{2}}+\rho_{\frac{3}{2}}\right)\ \exp\left(c_1^2\right)]\right\}^d$$

where $\zeta_q = q2^{2-1/q}\sqrt[q]{a_{1,q}}$ *and* $\rho_q = q2^{2-1/q}$.

Then the transition density of $p(s,x;t,y)$ *of diffusion*

$$dX_t = dW_t + b(X_t,t)dt$$

is bounded for all $|x-y| \le \sqrt{t-s} \le 1$ *by*

$$J_1(t-s)^{-\frac{d}{2}} \le p(s,x;t,y) \quad \le \quad J_2(t-s)^{-\frac{d}{2}} \tag{3.10}$$

Proof 1) The following results (3.11) and (3.12) are from [4]. Althought [4] only treated the homogeneous case, its proof also works for non-homogeneous case as the main method is comparison of drifts. Denote

$$\sigma(t,c) = \sqrt{\frac{1-e^{-2ct}}{2c}}$$

(set $\sigma(t,0)=\sqrt{t}$.) Then the lower bound is given by

$$\begin{aligned} & p(s,x;t,y) \\ \ge\ & \prod_{i=1}^d \left\{ h_{-c_2}(x_i-y_i,t-s,0)\exp\left[-2c_1\left(a_{1,1}\sqrt{\frac{e^{2c_2(t-s)}-1}{2c_2}}+|x_i-y_i|\right)-\frac{c_1^2}{2}(t-s)\right]\right\} \end{aligned} \tag{3.11}$$

for all $t>0$ and x,y, where

$$h_\alpha(x_i,t,z_i) = \frac{1}{\sqrt{2\pi}\sigma(t,\alpha)}\exp\left(-\frac{|z_i-e^{-\alpha t}x_i|^2}{2\sigma(t,\alpha)^2}\right).$$

The upper bound is given for every $q>1$ we have

$$\begin{aligned} p(s,x;t,y) \quad \le \quad & \prod_{i=1}^d \left\{ h_{c_2}(x_i-y_i,t-s,0) + \frac{c_1}{\sqrt{2\pi}\sigma(t-s,c_2)}\left(\zeta_q\sqrt{\frac{1-e^{-2c_2(t-s)}}{2c_2}}+\rho_q|x_i-y_i|\right)\right. \\ & \left. \times\exp\left(-\frac{e^{-2c_2(t-s)}|x_i-y_i|^2}{2q\sigma(t-s,c_2)^2}+\frac{c_1^2}{2(q-1)}(t-s)\right)\right\} \end{aligned} \tag{3.12}$$

2) When $|x_i-y_i| \le \sqrt{t-s}$, we get from (3.11) and (3.12)

$$\begin{aligned} & \prod_{i=1}^d \left\{ h_{-c_2}(x_i-y_i,t-s,0)\exp\left[-2c_1\left(a_{1,1}\sqrt{\frac{e^{2c_2(t-s)}-1}{2c_2}}+\sqrt{t-s}\right)-\frac{c_1^2}{2}(t-s)\right]\right\} \le p(s,x;t,y) \\ \le\ & \prod_{i=1}^d \left\{ h_{c_2}(x_i-y_i,t-s,0) + \frac{c_1}{\sqrt{2\pi}\sigma(t-s,c_2)}\left(\zeta_q\sqrt{\frac{1-e^{-2c_2(t-s)}}{2c_2}}+\rho_q\sqrt{t-s}\right)\ \exp\left(\frac{c_1^2}{2(q-1)}(t-s)\right)\right\} \end{aligned}$$

Furthermore, when $0 \le t-s \le 1$ and $q=\frac{3}{2}$,

$$
\begin{aligned}
& \prod_{i=1}^{d}\left\{h_{-c_2}(x_i-y_i,t-s,0)\exp\left[-2c_1\left(a_{1,1}\sqrt{\frac{e^{2c_2 t-s}-1}{2c_2}}+\sqrt{t-s}\right)-\frac{c_1^2}{2}(t-s)\right]\right\} \\
\ge\ & \left\{\frac{1}{\sqrt{2\pi}\sigma(t-s,-c_2)}\exp\left(-\frac{e^{2c_2(t-s)}(t-s)}{\frac{e^{2c_2(t-s)}-1}{c_2}}\right)\exp\left[-2c_1\left(a_{1,1}\sqrt{\frac{e^{2c_2(t-s)}-1}{2c_2}}+\sqrt{t-s}\right)-\frac{c_1^2}{2}(t-s)\right]\right\}^d \\
\ge\ & \left\{\frac{1}{\sqrt{2\pi}\sigma(t-s,-c_2)}\exp\left(-\frac{e^{2c_2(t-s)}(t-s)}{2(t-s)}\right)\exp\left[-2c_1\left(a_{1,1}\sqrt{\frac{e^{2c_2(t-s)}-1}{2c_2}}+\sqrt{t-s}\right)-\frac{c_1^2}{2}(t-s)\right]\right\}^d \\
\ge\ & \left\{\frac{1}{\sqrt{2\pi}\sigma(t-s,-c_2)}\exp\left(-\frac{1}{2}e^{2c_2}\right)\exp\left[-2c_1\left(a_{1,1}\sqrt{\frac{e^{2c_2}-1}{2c_2}}+1\right)-\frac{c_1^2}{2}\right]\right\}^d \\
\ge\ & (t-s)^{-\frac{d}{2}}\left\{\frac{1}{e^{2c_2}\sqrt{2\pi}}\exp\left(-\frac{1}{2}e^{2c_2}\right)\exp\left[-2c_1\left(a_{1,1}\sqrt{\frac{e^{2c_2}-1}{2c_2}}+1\right)-\frac{c_1^2}{2}\right]\right\}^d \qquad (3.13)
\end{aligned}
$$

and

$$
\begin{aligned}
& \prod_{i=1}^{n}\left\{h_{c_2}(x_i-y_i,t-s,0)+\frac{c_1}{\sqrt{2\pi}\sigma(t-s,c_2)}\left(\zeta_q\sqrt{\frac{1-e^{-2c_2(t-s)}}{2c_2}}+\rho_q\sqrt{t-s}\right)\exp\left(\frac{c_1^2}{2(q-1)}(t-s)\right)\right\} \\
\le\ & \left\{\frac{1}{\sqrt{2\pi}\sigma(t-s,c_2)}+\frac{c_1}{\sqrt{2\pi}\sigma(t-s,c_2)}\left(\zeta_q+\rho_q\right)\exp\left(\frac{c_1^2}{2(q-1)}\right)\right\}^d \\
\le\ & (t-s)^{-\frac{d}{2}}\left\{\frac{e^{2c_2}}{\sqrt{2\pi}}[1+c_1\left(\zeta_{\frac{3}{2}}+\rho_{\frac{3}{2}}\right)\exp\left(c_1^2\right)]\right\}^d \qquad (3.14)
\end{aligned}
$$

where we used the fact that $\frac{1-e^{-2c_2(t-s)}}{2c_2(t-s)} \ge e^{-2c_2}$ in the last inequality.

Theorem 3.6 *Define* $\forall t \ge s$,

$$z_{\theta,x,s,t} = x + \int_s^t \mu(z_{\theta,x,s,u},u,\theta)du.$$

When $|z_{\theta,x,s,t}-y| \le \sqrt{t-s} \le 1$,

$$K_6(t-s)^{-\frac{d}{2}} \le p_\theta(s,x;t,y) \le K_7(t-s)^{-\frac{d}{2}}$$

where

$$K_6 = \{\frac{1}{e^{2\sqrt{d}K_2}\sqrt{2\pi}}\exp(-\frac{1}{2}e^{2K_2})\exp[-2\sqrt{d}K_2(\sqrt{\frac{2}{\pi}}\sqrt{\frac{e^{2\sqrt{d}K_2}-1}{2\sqrt{d}K_2}}+1)-\frac{(\sqrt{d}K_2)^2}{2}]\}^d < 1$$

and

$$K_7 = \left\{\frac{e^{2\sqrt{d}K_2}}{\sqrt{2\pi}}[1+\sqrt{d}K_2\left(\zeta_{\frac{3}{2}}+\rho_{\frac{3}{2}}\right)\exp\left((\sqrt{d}K_2)^2\right)]\right\}^d$$

Proof Let

$$X_t = x + W_t + \int_s^t \mu(X_u, u, \theta) du$$

and let $Z_t = X_t - z_{\theta,x,s,t}$ $(\forall t \geq s)$. Then

$$\begin{aligned} dZ_t &= dW_t + \mu(X_t, t, \theta)dt - \mu(z_{\theta,x,s,t}, t, \theta)dt \\ &= dW_t + b(Z_t, t, \theta)dt \end{aligned} \tag{3.15}$$

where $b(x', t, \theta) = \mu(x' + z_{\theta,x,s,t}, t, \theta) - \mu(z_{\theta,x,s,t}, t, \theta)$. If $|y - z_{\theta,x,s,t}| \leq 1$, then

$$|b(x', t, \theta)| \leq K_2|x'| \leq K_2(1 + |x' - (y - z_{\theta,x,s,t})|).$$

Thus we can apply the previous Lemma to $P[X_t \in dy \; |X_s = x] = P[Z_t \in d(y - z_{\theta,x,s,t}) \; |Z_s = 0]$ and get the desired result for $c_1 = c_2 = \sqrt{d}K_2$ and $a_{1,1} = \sqrt{\frac{2}{\pi}}$.

4 The key lemma

Lemma 4.1

$$\begin{aligned} & E_\theta^T[\exp\{-\frac{k-2}{k^2}\int_0^T [\mu(X_t, t, \theta + u) - \mu(X_t, t, \theta)]^2 dt\}] \\ \leq \; & \left\{1 - (1 \wedge (K_4K_6))(\frac{1}{2})^{\frac{2+d}{2}}\left(1 - \exp\{-\frac{K_3^4}{2}\frac{(k-2)}{k^2}u^2\}\right)\right\}^{[\frac{T}{K_3^2}]} \end{aligned} \tag{4.16}$$

where $[\frac{T}{K_3^2}]$ *is the integer part of* $\frac{T}{K_3^2}$.

Proof 1) By Jensen's inequality, for each pair $s_1 < s_2$,

$$\begin{aligned} & E_\theta^T[\exp\{-\frac{k-2}{k^2}\int_{s_1}^{s_2} [\mu(X_t, t, \theta + u) - \mu(X_t, t, \theta)]^2 dt\}|X_{s_1}] \\ \leq \; & E_\theta^T[\frac{1}{s_2 - s_1}\int_{s_1}^{s_2} \exp\{-(s_2 - s_1)\frac{k-2}{k^2}[\mu(X_t, t, \theta + u) - \mu(X_t, t, \theta)]^2\}dt|X_{s_1}] \\ = \; & \frac{1}{s_2 - s_1}\int_{s_1}^{s_2}\{\int \exp\{-(s_2 - s_1)\frac{k-2}{k^2}[\mu(y, t, \theta + u) - \mu(y, t, \theta)]^2\} \\ & p_\theta(s_1, X_{s_1}; t, y)dy\}dt \end{aligned} \tag{4.17}$$

Denote $A_t = \{y; \; |\mu(y, t, \theta + u) - \mu(y, t, \theta)| < \sqrt{\frac{s_2 - s_1}{2}}|u|\}$, then we have easily

$$\begin{aligned} & \int \exp\{-(s_2 - s_1)\frac{k-2}{k^2}[\mu(y, t, \theta + u) - \mu(y, t, \theta)]^2\}p_\theta(s_1, X_{s_1}; t, y)dy \\ = \; & \int \chi_{A_t} \exp\{-(s_2 - s_1)\frac{k-2}{k^2}[\mu(y, t, \theta + u) - \mu(y, t, \theta)]^2\}p_\theta(s_1, X_{s_1}; t, y)dy \\ & + \int \chi_{A_t^c} \exp\{-(s_2 - s_1)\frac{k-2}{k^2}[\mu(y, t, \theta + u) - \mu(y, t, \theta)]^2\}p_\theta(s_1, X_{s_1}; t, y)dy \end{aligned}$$

$$\begin{aligned}
&\le \int \chi_{A_t} p_\theta(s_1, X_{s_1}; t, y)dy \\
&\quad + \int \chi_{A_t^c} \exp\{-(s_2-s_1)\frac{k-2}{k^2}[\frac{s_2-s_1}{2}]u^2\} p_\theta(s_1, X_{s_1}; t, y)dy \\
&= 1 - \int \chi_{A_t^c} \left(1 - \exp\{-(s_2-s_1)\frac{k-2}{k^2}[\frac{s_2-s_1}{2}]u^2\}\right) p_\theta(s_1, X_{s_1}; t, y)dy \\
&\le 1 - \int \chi_{A_t^c} \chi_{\{|z_{\theta,(X_{s_1}),s_1,t} - y| \le \sqrt{\frac{s_2-s_1}{2}}\}} \left(1 - \exp\{-(s_2-s_1)\frac{k-2}{k^2}[\frac{s_2-s_1}{2}]u^2\}\right) p_\theta(s_1, X_{s_1}; t, y)dy
\end{aligned}$$

By Theorem 3.6, when $\frac{s_2-s_1}{2} \le t - s_1 \le s_2 - s_1 \le 1$ and $|z_{x,s_1,t} - y| \le \sqrt{\frac{s_2-s_1}{2}}$,

$$K_6(s_2-s_1)^{-\frac{d}{2}} \le p_\theta(s_1, x; t, y).$$

Thus when $\sqrt{\frac{s_2-s_1}{2}} \le K_3$ and $\frac{s_2-s_1}{2} \le t - s_1 \le s_2 - s_1 \le 1$,

$$\begin{aligned}
&\int \exp\{-(s_2-s_1)\frac{k-2}{k^2}[\mu(y,t,\theta+u) - \mu(y,t,\theta)]^2\} p_\theta(s_1, X_{s_1}, t, y)dy \\
&\le 1 - \int \chi_{A^c} \chi_{\{|z_{X_{s_1},s_1,t} - y| \le \sqrt{\frac{s_2-s_1}{2}}\}} \left(1 - \exp\{-\frac{(s_2-s_1)^2}{2}\frac{(k-2)}{k^2}u^2\}\right) K_6(s_2-s_1)^{-\frac{d}{2}}dy \\
&\le 1 - K_4(\frac{1}{2})^{\frac{d}{2}} \left(1 - \exp\{-\frac{(s_2-s_1)^2}{2}\frac{(k-2)}{k^2}u^2\}\right) K_6 \qquad (4.18)
\end{aligned}$$

where we used Hypothesis (ii) in the last inequality. (4.17) and (4.18) yield

$$\begin{aligned}
& E_\theta^T[\exp\{-\frac{k-2}{k^2}\int_{s_1}^{s_2}[\mu(X_t,t,\theta+u) - \mu(X_t,t,\theta)]^2 dt\}|X_{s_1}] \\
\le\; & \frac{1}{s_2-s_1}\int_{s_1}^{\frac{s_1+s_2}{2}} \{\int \exp\{-(s_2-s_1)\frac{k-2}{k^2}[\mu(y,t,\theta+u) - \mu(y,t,\theta)]^2\} \\
& p_\theta(s_1, X_{s_1}; t, y)dy\}dt \\
& + \frac{1}{s_2-s_1}\int_{\frac{s_1+s_2}{2}}^{s_2} \{\int \exp\{-(s_2-s_1)\frac{k-2}{k^2}[\mu(y,t,\theta+u) - \mu(y,t,\theta)]^2\} \\
& p_\theta(s_1, X_{s_1}; t, y)dy\}dt \\
\le\; & 1 - K_4K_6(\frac{1}{2})^{\frac{2+d}{2}} \left(1 - \exp\{-2K_3^4\frac{(k-2)}{k^2}u^2\}\right) \qquad (4.19)
\end{aligned}$$

2) Now

$$\begin{aligned}
& E_\theta^T[\exp\{-\frac{k-2}{k^2}\int_0^T [\mu(X_t,t,\theta+uT^{-\frac{1}{2}}) - \mu(X_t,t,\theta)]^2 dt\}] \\
\le\; & E_\theta^T[\exp\{-\frac{k-2}{k^2}\sum_{i=0}^{[\frac{T}{K_3^2}]-1}\int_{iK_3^2}^{(i+1)K_3^2}[\mu(X_t,t,\theta+uT^{-\frac{1}{2}}) - \mu(X_t,t,\theta)]^2 dt\}] \\
=\; & E_\theta^T\{\exp\{-\frac{k-2}{k^2}\sum_{i=0}^{[\frac{T}{K_3^2}]-2}\int_{iK_3^2}^{(i+1)K_3^2}[\mu(X_t,t,\theta+uT^{-\frac{1}{2}}) - \mu(X_t,t,\theta)]^2 dt\}]
\end{aligned}$$

$$E_\theta^T[\exp\{-\frac{k-2}{k^2}\int_{K_3^2([\frac{T}{K_3^2}]-1)}^{K_3^2[\frac{T}{K_3^2}]}[\mu(X_t,t,\theta+uT^{-\frac{1}{2}})-\mu(X_t,t,\theta)]^2dt\}|X_{K_3^2([\frac{T}{K_3^2}]-1)}]\} \quad (4.20)$$

(4.20) and (4.19) yield

$$\begin{aligned} & E_\theta^T[\exp\{-\frac{1}{3}\int_0^T[\mu(X_t,t,\theta+uT^{-\frac{1}{2}})-\mu(X_t,t,\theta)]^2dt\}] \\ \le\ & E_\theta^T\{\exp\{-\frac{k-2}{k}\sum_{i=0}^{[\frac{T}{K_3^2}]-2}\int_{([\frac{T}{K_3^2}]-1)K_3^2}^{[\frac{T}{K_3^2}]K_3^2}[\mu(X_t,t,\theta+uT^{-\frac{1}{2}})-\mu(X_t,t,\theta)]^2dt\}]\} \\ & \{1-K_4K_6(\frac{1}{2})^{\frac{2+d}{2}}\left(1-\exp\{-\frac{K_3^4}{2}\frac{(k-2)}{k^2}u^2\}\right)\} \end{aligned} \quad (4.21)$$

Repeatedly applying (4.21) to each i, we get

$$\begin{aligned} & E_\theta^T[\exp\{-\frac{k-2}{k^2}\int_0^T[\mu(X_t,t,\theta+uT^{-\frac{1}{2}})-\mu(X_t,t,\theta)]^2dt\}] \\ \le\ & \left\{1-K_4K_6(\frac{1}{2})^{\frac{2+d}{2}}\left(1-\exp\{-\frac{K_3^4}{2}\frac{(k-2)}{k^2}u^2\}\right)\right\}^{[\frac{T}{K_3^2}]} \end{aligned}$$

Moreover, from the definition, it is easy to see that $K_4 \le 1$ and $K_6 < 1$. Thus we get the conclusion.

5 Proof of the main theorem

Since $M_T(0) = 1$, it is easy to see that

$$P_\theta^T\{|\hat\theta_T-\theta|>r\} \quad \le \quad P_\theta^T\{\sup_{|u|\ge r}M_T(u)\ge 1\}. \quad (5.22)$$

Denote by $Z^{\tilde d}$ the set of $\tilde d$ dimemsional integers. Take $k=\tilde d+2$,

$$\begin{aligned} & P_\theta^T\{\sup_{|u|\ge r}M_T(u)\ge 1\} \\ \le\ & \sum_{\{j\in Z^{\tilde d};\ \frac{1}{n\sqrt{T}}j+\theta\in\Theta,\ |\frac{1}{n\sqrt{T}}j|\ge r\}} P_\theta^T[M_T^{\frac{1}{k}}(\frac{1}{n\sqrt{T}}j)\ge\frac{1}{2}]+P_\theta^T\{\sup_{|u-v|\le\frac{\sqrt{\tilde d}}{n\sqrt{T}}}|M_T^{\frac{1}{k}}(u)-M_T^{\frac{1}{k}}(v)|\ge\frac{1}{2}\} \\ \le\ & 2\sum_{\{j\in Z^{\tilde d};\ \frac{1}{n\sqrt{T}}j+\theta\in\Theta,\ |\frac{1}{n\sqrt{T}}j|\ge\rho\}} E_\theta^T[M_T^{\frac{1}{k}}(\frac{1}{n\sqrt{T}}j)] \\ & +2^{\tilde d+2}K_5[2^{\tilde d+1}(\frac{\tilde d+1}{2(\tilde d+2)^2})^{\tilde d+2}K_0^{\tilde d+2}K_1^{2(\tilde d+2)}T^{\tilde d+2}+2^{\tilde d+1}\frac{1}{(\tilde d+2)^{\tilde d+2}}a_{\tilde d+2}K_1^{\tilde d+2}T^{\frac{\tilde d+2}{2}}]\frac{\sqrt{\tilde d}}{n\sqrt{T}} \\ \le\ & 2K_0^{\tilde d}n^{\tilde d}T^{\frac{\tilde d}{2}}\left\{1-(1\wedge(K_4K_6))(\frac{1}{2})^{\frac{2+d}{2}}\left(1-\exp\{-\frac{K_3^4}{2}\frac{\tilde d}{(\tilde d+2)^2}r^2\}\right)\right\}^{\frac{1}{2}[\frac{T}{K_3^2}]} \\ & +2^{\tilde d+2}K_5[2^{\tilde d+1}(\frac{\tilde d+1}{2(\tilde d+2)^2})^{\tilde d+2}K_0^{\tilde d+2}K_1^{2(\tilde d+2)}+2^{\tilde d+1}\frac{1}{(\tilde d+2)^{\tilde d+2}}a_{\tilde d+2}K_1^{\tilde d+2}T^{\frac{-2-\tilde d}{2}}]n^{-1}T^{(\tilde d+2)-\frac{1}{2}}\tilde d^{\frac{1}{2}} \end{aligned}$$

where the second inequality is by Chebeshev's inequality and Lemma 3.3 with $h = \frac{\sqrt{\tilde{d}}}{n\sqrt{T}}$ and $\beta = \frac{1}{\tilde{d}+2}$. The third one is by Lemmas 3.4 and 4.1. Take n as the integer part of the following value

$$\left[\left\{ 1 - (1 \wedge (K_4 K_6))(\frac{1}{2})^{\frac{2+d}{2}} \left(1 - \exp\{-\frac{K_3^4}{2} \frac{\tilde{d}}{(\tilde{d}+2)^2} r^2\} \right) \right\}^{-\frac{1}{4\tilde{d}}[\frac{T}{K_3^2}]} \right],$$

which is larger or equal to 1 ($\forall r > 0$). When $T \geq 1$,

$$\begin{aligned}
& P_\theta^T\{\sup_{|u|\geq r} M_T(u) \geq 1\} \\
\leq\ & (2K_0^{\tilde{d}} T^{\frac{\tilde{d}}{2}} + K_8 T^{(\tilde{d}+2)-\frac{1}{2}}) \left\{ 1 - (1 \wedge (K_4 K_6))(\frac{1}{2})^{\frac{2+d}{2}} \left(1 - \exp\{-\frac{K_3^4}{2} \frac{\tilde{d}}{(\tilde{d}+2)^2} r^2\} \right) \right\}^{\frac{1}{4\tilde{d}}[\frac{T}{K_3^2}]} \\
\leq\ & (2K_0^{\tilde{d}} T^{\frac{\tilde{d}}{2}} + K_8 T^{(\tilde{d}+2)-\frac{1}{2}}) \left\{ 1 - (1 \wedge (K_4 K_6))(\frac{1}{2})^{\frac{2+d}{2}} \left(\frac{K_3^4}{2} \frac{\tilde{d}}{(\tilde{d}+2)^2} r^2 \exp\{-\frac{K_3^4}{2} \frac{\tilde{d}}{(\tilde{d}+2)^2} r^2\} \right) \right\}^{\frac{1}{4\tilde{d}}[\frac{T}{K_3^2}]} \\
\leq\ & (2K_0^{\tilde{d}} T^{\frac{\tilde{d}}{2}} + K_8 T^{(\tilde{d}+2)-\frac{1}{2}}) \left\{ 1 - (1 \wedge (K_4 K_6))(\frac{1}{2})^{\frac{2+d}{2}} \frac{K_3^4}{2} \frac{\tilde{d}}{(\tilde{d}+2)^2} \exp\{-\frac{K_0^2 K_3^4}{2} \frac{\tilde{d}}{(\tilde{d}+2)^2}\} r^2 \right\}^{\frac{1}{4\tilde{d}}[\frac{T}{K_3^2}]} \\
\leq\ & (2K_0^{\tilde{d}} T^{\frac{\tilde{d}}{2}} + K_8 T^{(\tilde{d}+2)-\frac{1}{2}}) \left\{ 1 - K_9 r^2 \right\}^{\frac{1}{4\tilde{d}}[\frac{T}{K_3^2}]}
\end{aligned}$$

Since $K_0^2 y e^{-K_0^2 y} \leq e^{-1}$, it is easy to check

$$K_9 = (1 \wedge (K_4 K_6))(\frac{1}{2})^{\frac{2+d}{2}} \frac{K_3^4}{2} \frac{\tilde{d}}{(\tilde{d}+2)^2} \exp\{-\frac{K_0^2 K_3^4}{2} \frac{\tilde{d}}{(\tilde{d}+2)^2}\} \leq \frac{1}{2^{\frac{2+d}{2}} e K_0^2}$$

Thus we get Theorem 2.1.

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